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The Decision to Lever

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Abstract

We present a performance attribution method for levered strategies. This method provides a framework for deciding whether and how much to lever. Essential elements are forecasts for the expected return and volatility of the fully-invested *source portfolio* that is to be levered, borrowing rates, the covariance of strategy leverage with the excess return to the source portfolio over the borrowing rate, compounding effects and trading costs. In an empirical study, we show these elements provide a complete explanation of the realized returns of three levered low-risk strategies. For one of the strategies, market frictions cut friction-free cumulative returns over an 84-year period by a factor of 1000.

Key terms: Leverage, source portfolio, transaction costs, borrowing excess return, trading costs, low-risk portfolio, fixed leverage, conditional volatility target, unconditional volatility target

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1 Introduction

Even among the most conservative and highly regulated investors such as US public pension funds, the use of levered investment strategies is widespread and growing.[1] Initially, these plans invested primarily in highly-rated bonds, without leverage. Subsequently, they added equity to their portfolios, but still without leverage. More recently, they have added real estate, commodities, private equity, and alternative investments, some of which (especially real estate and hedge funds) involve leverage. In the period since the financial crisis, strategies such as risk parity that lever holdings of publicly traded securities have emerged as candidates for these investment portfolios.

In a friction-free world, where both excess return and volatility scale linearly with leverage, the decision to invest in a levered strategy is already subtle. In our friction-rich world, leverage is entangled with borrowing and trading costs as well as management fees and market impact, which lead to complex investment questions. For example:

- How does leverage affect a strategy's realized return?
- How do levered strategies perform in different economic climates?
- What factors are relevant to the decision to lever?
- How much leverage is optimal?

To address these questions we show that the return to a levered strategy can be attributed completely to simpler elements that an investor may be able to measure or forecast or at least contemplate. The elements include the fully invested *source* portfolio that is to be levered, the leverage, the borrowing rate and the trading costs. We show that these elements provide a complete attribution of realization of return of the levered strategy.

1.1 Leverage and Market Frictions

Our focus in this paper is on the effect of leverage in the presence of market frictions. This leads to a recipe for an investor with given beliefs about the distribution of future asset returns to decide whether to lever, and if so, how; it applies whether or not the investor's beliefs are correct.

In a friction-free world, excess return to a levered portfolio in a single period is the product of the leverage and the excess return of the source portfolio over the risk-free rate, which is also the borrowing rate. This means that if the source portfolio excess return is positive, more leverage means more excess return.

In our friction-rich world, the borrowing rate[2] exceeds the risk-free rate, and single period return to the levered strategy is driven by the product of leverage and the expected

[1] See, for example, Kozlowski (2013).

[2] Leverage may be achieved through explicit borrowing. It may also be achieved through the use of derivative contracts, such as futures. In these derivative contracts, the borrowing cost is implicit rather

borrowing excess return, which is the return to the source in excess of the borrowing rate. In other words, there is a higher threshold for leverage to have a positive impact in a friction-rich world than in a friction-free world: the return to the source portfolio must exceed the borrowing rate.

Over a long horizon, the return to a levered strategy depends on the co-movement of leverage and borrowing excess return. If the correlation between leverage and borrowing excess return to the source portfolio is positive, all else equal, more volatility (for leverage and for the source portfolio) means more excess return. In our friction-rich world, it is again, the borrowing excess return that counts in deciding whether or not to lever.

The use of leverage results in turnover for two reasons. First, movement in the value of the source portfolio changes the leverage; rebalancing is needed to maintain even a fixed leverage target. Second, a leverage rule may call for leverage to change over time, in response to market conditions, and this requires additional turnover.

These facts are summarized in Formula (9):

$$\begin{aligned} E\left[r^{\mathbf{L}}-r^{\mathbf{S}}\right] &= E\left[(\lambda-1)\left(r^{\mathbf{S}}-r^{b}\right)\right] \\ &= E\left[\lambda-1\right]E\left[r^{\mathbf{S}}-r^{b}\right]+\operatorname{cov}(\lambda, r^{\mathbf{S}}-r^{b})-E\left[r^{\mathbf{TC}}\right]. \end{aligned}$$

This says that the expected return of the levered strategy minus the expected return of the source portfolio is the product of the expectation of the leverage minus one and the expected borrowing excess return of the source portfolio, plus the covariance of the leverage and the borrowing excess return of the source portfolio, minus the expected value of trading costs needed to implement the leverage rule.

Finally, the sample mean of realized arithmetic returns does not correctly account for compounding. The compounding correction, which is given in Formula (11), involves expected returns and variances of the source and levered strategies. Specifically, we consider the difference in *log geometric average*, $\log(1+G)$, of a levered strategy and its source:

$$\begin{aligned} &\log\left(1+G\left[r^{\mathbf{L}}\right]\right)-\log\left(1+G\left[r^{\mathbf{S}}\right]\right) \\ &\sim\ E\left[r^{\mathbf{L}}-r^{\mathbf{S}}\right]\left(1-\frac{E\left[r^{\mathbf{L}}\right]+E\left[r^{\mathbf{S}}\right]}{2}\right)-\frac{\operatorname{var}\left(r^{\mathbf{L}}\right)-\operatorname{var}\left(r^{\mathbf{S}}\right)}{2}, \end{aligned}$$

where G denotes the geometric average. Ranking by log geometric average is identical to ranking by cumulative return. This formula says that to properly rank levered strategies by excess cumulative return over their sources, the expected return of the levered strategy minus the expected return of the source portfolio must be scaled and then corrected for variance. For a given source strategy, the variance correction is quadratic in the leverage and it may reverse the apparent performance of two levered strategies.

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than explicit, but it is real and is at a rate higher than the T-Bill rate. For example, Naranjo (2009) find that the implicit borrowing cost using futures is approximately the applicable LIBOR rate, applied to the notional value of the futures contract.

1.2 Empirical Studies

We demonstrate the efficacy of our ideas by attributing the difference in expected excess return of three levered strategies over a common source portfolio consisting of US treasury bonds. The analysis is based on Formulas (9) and (11). One of our strategies explicitly specifies a constant leverage. The other two are based on volatility targeting, and their leverage is a consequence of the estimated volatilities of the source portfolio and target volatility. Since they have a common source portfolio, the only difference between these strategies is the timing and level of their leverage. Consequently, the degree to which these three similar-sounding strategies perform differently in three different time periods, and before and after transaction costs is both remarkable and unsettling. In the volatility targeting strategies, it is the implicit leverage and not the explicitly specified volatility that is the key to understanding strategy performance. We provide realistic, empirical examples of the following phenomena:

- The covariance between leverage and borrowing excess return can make a substantial contribution to the performance of volatility targeting strategies, both before and after accounting for transaction costs. The contribution can be either positive or negative.
- Even a linear model of transaction costs leads to a material drag on performance in a levered strategy.
- Strategy performance depends on the start and end dates of the study period.
- Arithmetic and geometric averages can produce different rankings of leverage rules. The leverage rule with the higher geometric ranking will produce greater value at the end of the period.

1.3 Outline of this Paper

Section 2 decomposes the return of a levered strategy into simpler constituents. This decomposition can be used to support decisions about when and how to lever. Section 3 describes various kinds of levered low-risk strategies and shows that our decomposition provides a complete explanation of how these strategies have performed historically. Section 4 sets out a recipe for deciding how to choose a source portfolio and whether and how to lever it. Section 5 discusses the rationale behind the parameter settings in our empirical strategies. Section 6 concludes.

Four appendices supplement the main narrative. Appendix A provides a brief survey of the literature. Appendix B gives a detailed description of our data sources. Appendix C outlines a novel methodology for determining turnover due to leverage. In this article, we use a linear model to estimate the cost of leverage-induced turnover. Our model neglects market impact; nevertheless, leverage-induced turnover is quite expensive. Appendix D describes our derivation of Formula (11), which relates geometric return to arithmetic return. It is the former, and not the latter, that reliably ranks strategies by cumulative return over multiple periods.

2 Performance Attribution of Levered Strategies

A necessary condition for levering a source portfolio is that the expected return of the levered strategy exceed the expected return of its source portfolio. An analysis of this condition depends on an understanding of how leverage affects return in the presence of market frictions. We map out that analysis here.

A *levered strategy* is built from a fully invested *source portfolio* and a *leverage rule*. An investor has a certain amount of capital, L. He chooses a leverage ratio λ, borrows $(\lambda - 1)L$, and invests λL in the source portfolio. In what follows, we assume $\lambda > 1$.

2.1 Attribution of Arithmetic Return

In a friction-free world, the relationship between the single-period returns to a levered portfolio, $r^{\mathbf{L}}$, and to its source portfolio, $r^{\mathbf{S}}$, is given by:

$$r^{\mathbf{L}} = \lambda r^{\mathbf{S}} - (\lambda - 1) r^f, \tag{1}$$

where r^f is the risk-free rate. It follows that:

$$r^{\mathbf{L}} - r^{\mathbf{S}} = (\lambda - 1)\left(r^{\mathbf{S}} - r^f\right). \tag{2}$$

Since $\lambda > 1$, Formula (2) shows that the levered portfolio will outperform the source portfolio when the source return exceeds the risk-free rate, but not otherwise. To evaluate a levered strategy over multiple periods, we take the expectation of Formula (2), which gives:

$$\begin{aligned} E\left[r^{\mathbf{L}} - r^{\mathbf{S}}\right] &= E\left[(\lambda - 1)\left(r^{\mathbf{S}} - r^f\right)\right] \\ &= E\left[\lambda - 1\right] E\left[r^{\mathbf{S}} - r^f\right] + \operatorname{cov}(\lambda, r^{\mathbf{S}} - r^f). \end{aligned} \tag{3}$$

Formula (3) shows that for the levered strategy to outperform its source strategy over a long horizon, it is necessary that either the expected excess return be positive or the correlation between the excess return and the leverage be positive, or both. If both conditions are met, then increasing leverage will increase expected return (and also increase the volatility of the strategy).

Note that we can interpret the expectation and covariance in Formula (3) in two ways: prospectively and retrospectively. Prospectively, they represent the expectation and covariance under the true probability distribution (or alternatively, under the investor's beliefs about the true probability distribution). Retrospectively, they represent the realized mean and realized covariance of the returns.[3]

In a friction-rich world where the borrowing rate, r^b, exceeds the risk-free rate, the relationship between the return to a levered portfolio and the return to its source is:

$$r^{\mathbf{L}} = \lambda r^{\mathbf{S}} - (\lambda - 1) r^b. \tag{4}$$

Formula (2) for the single-period excess return of a levered portfolio over its source portfolio is replaced with:

$$r^{\mathbf{L}} - r^{\mathbf{S}} = (\lambda - 1)\left(r^{\mathbf{S}} - r^b\right). \tag{5}$$

Note that the bar for leverage to have a positive impact has gotten higher: the borrowing excess return, $r^{\mathbf{S}} - r^b$, must be positive. Over multiple periods, Formula (3) is replaced with:

$$\begin{aligned} E\left[r^{\mathbf{L}} - r^{\mathbf{S}}\right] &= E\left[(\lambda - 1)\left(r^{\mathbf{S}} - r^b\right)\right] \\ &= E\left[\lambda - 1\right] E\left[r^{\mathbf{S}} - r^b\right] + \operatorname{cov}(\lambda, r^{\mathbf{S}} - r^b). \end{aligned} \tag{6}$$

Note that it is the borrowing excess return that is relevant to the decision about whether or not to lever in a friction-rich world.

The cost of trading that stems from leverage-induced turnover is a drag on the return to a levered strategy. To take account of this effect, we extend Formula (4):

$$r^{\mathbf{L}} = \lambda r^{\mathbf{S}} - (\lambda - 1) r^b - r^{\mathbf{TC}}, \tag{7}$$

where $r^{\mathbf{TC}}$ is the return that is sacrificed to leverage-induced turnover. This leads to an extension of Formula (5) for the single-period excess return of the levered portfolio over the source portfolio:

$$r^{\mathbf{L}} - r^{\mathbf{S}} = (\lambda - 1)\left(r^{\mathbf{S}} - r^b\right) - r^{\mathbf{TC}}. \tag{8}$$

and an extension of Formula (6) for the expected excess return:

$$\begin{aligned} E\left[r^{\mathbf{L}} - r^{\mathbf{S}}\right] &= E\left[(\lambda - 1)\left(r^{\mathbf{S}} - r^b\right)\right] \\ &= E\left[\lambda - 1\right] E\left[r^{\mathbf{S}} - r^b\right] + \operatorname{cov}(\lambda, r^{\mathbf{S}} - r^b) - E\left[r^{\mathbf{TC}}\right]. \end{aligned} \tag{9}$$

Any estimate of $r^{\mathbf{TC}}$ relies on assumptions about the relationship between turnover and trading cost. We assume cost depends linearly on the fraction of the portfolio that turns over, and we use Formula (22) to estimate $r^{\mathbf{TC}}$ in our empirical studies. More information is in Appendix C.

[3] Note that we take the realized covariance, obtained by dividing by the number of dates, rather than the realized sample covariance, which would be obtained by dividing by one less than the number of dates. The realized sample covariance is an unbiased estimator of the population covariance, but it is the realized covariance that makes Formula (3) true.

2.2 Attribution of Geometric Return

In some circumstances, Formula (9) provides a reasonable basis for analyzing the difference in cumulative return to a levered strategy and its source, and hence it informs the decision to lever. However, Formula (9) is based on arithmetic expected return, which does not properly account for compounding. The correction for compounding imposes a drag on compound return that is not captured in the average of arithmetic returns. If the levered strategy has high or unstable volatility, in response either to high or variable volatility of the source portfolio or to high or variable leverage, the drag may be substantial.

Thus, a more complete decision rule is based on the geometric expected return, which does properly account for compounding. If we have monthly returns for months $t = 0, 1 \ldots, T-1$ the geometric average of the monthly returns is:

$$G[r] = \left(\prod_{t=0}^{T-1} (1 + r_t) \right)^{1/T} - 1 \tag{10}$$

where r_t is the arithmetic return in month t. Given two strategies, the one with the higher geometric average will outperform the other, taking compounding into account. Since the logarithm is strictly increasing, we obtain exactly the same ranking if we use the *log geometric average* given by $\log(1 + G[r])$ instead. The advantage of using $\log(1 + G[r])$ rather than $G[r]$ is that it has a simpler relationship to arithmetic expected return. In Appendix 2.2, we show that the following holds to a high degree of approximation:

$$\begin{aligned} &\log\left(1 + G\left[r^{\mathbf{L}}\right]\right) - \log\left(1 + G\left[r^{\mathbf{S}}\right]\right) \\ \sim \quad & E\left[r^{\mathbf{L}} - r^{\mathbf{S}}\right] \left(1 - \frac{E\left[r^{\mathbf{L}}\right] + E\left[r^{\mathbf{S}}\right]}{2}\right) - \frac{\operatorname{var}\left(r^{\mathbf{L}}\right) - \operatorname{var}\left(r^{\mathbf{S}}\right)}{2} \end{aligned} \tag{11}$$

Geometric decision rules based on Formula (10) always produce the correct ranking, taking compounding into account. The approximation in Formula (11) is more convenient and should in practice make little difference in ranking. Since the compounding correction involves a drag on return that increases with leverage, geometric decision rules will generally give more stringent criteria for levering than arithmetic rules.

3 Empirical Studies

An investor may lever in connection with specified investment rules. In what follows, we discuss three levered strategies. The first explicitly specifies constant leverage, while leverage is implicitly determined by volatility levels in the second and third.

3.1 Three Levered Strategies

The simplest strategy we consider targets a *fixed leverage*, and it is denoted FLT. If $\lambda < 1$, the fraction $1 - \lambda$ of portfolio value is invested in a risk-free asset with interest rate r^f

and the constant leverage strategy is contrarian. This means that a decline in the value of the source portfolio prompts buying, while a rise in the value of the source portfolio prompts selling. If $\lambda > 1$, then $1 - \lambda$ times the equity in the strategy must be financed at a borrowing rate $r^b \geq r^f$. In this situation, constant leverage is a momentum strategy, in which a decline in the value of the source portfolio prompts selling, while a rise in the value of the source portfolio prompts buying. The borrowing rate r^b may be explicit, as for a margin account, or it may be implicit, for example, when leverage is obtained through derivatives such as options or futures contracts. If the source portfolio is low-risk, then an investor would typically choose $\lambda > 1$ in order to achieve higher expected returns than the unlevered low-risk portfolio could provide.

An alternative to constant leverage is *volatility targeting*, in which leverage is adjusted up or down so that the estimated volatility of the strategy follows a target. We consider two versions of this strategy. Volatility targeting is *unconditional* if the volatility target is a fixed number; at each rebalancing, leverage is adjusted so that the estimated volatility of the strategy matches the fixed target. A volatility targeting strategy is *conditional* if the volatility target is a sequence of numbers obtained by estimating the volatility of a *target strategy*; at each rebalancing, leverage is adjusted so that the estimated volatility of the strategy matches the estimated volatility of the target strategy. Our unconditional and conditional volatility strategies are denoted UVT and CVT. If the source portfolio is low-risk, then an investor would typically choose an unconditional volatility target greater than the volatility of the source portfolio, or a conditional volatility target strategy that is more volatile than the source portfolio, in order to achieve higher expected returns than the unlevered low-risk portfolio can provide.

An investor considering a levered strategy may want to think carefully about the different choices outline above. All else equal, these similar-sounding strategies respond differently to changing market conditions, and all three strategies call for additional investment in the source portfolio when its price rises. The details are in Table 1.

Table 1: Strategy Responses to Changes in Market Conditions

Trigger	Response: FLT	UVT	CVT
Increase in Target Volatility	no change	no change	$\uparrow$ leverage
Increase in Source Volatility	no change	$\downarrow$ leverage	$\downarrow$ leverage
Increase in Price of Source	buy source	buy source	buy source

Table 1: Responses of levered strategies to changes in market conditions.

We implement the three leverage rules just described when the source portfolio is a US treasury bond index and the target is a US equity index. Strategies are rebalanced monthly and volatility forecasts used to determine strategy weights are based on a 36-month trailing estimate.

3.2 Performance Attribution of Three Levered Strategies

We carry out the performance attribution described in Formula (9) for each of our three levered strategies, FLT, UVT and CVT. We first consider an 84-year long sample which begins in January 1929 and ends in December 2012. We also consider two subperiods. The first is the 20-year period beginning in January 1961 and ending in December 1980; this period was marked by variable volatility that resulted in erratic leverage patterns for UVT and CVT. The second is the 32-year period beginning in January 1981 and ending in December 2012; the leverage in both UVT and CVT was relatively low and relatively stable. In a base case that assumes no market frictions, the borrowing rate is taken to be the T-Bill rate and trading is costless. In a more realistic setting, the borrowing rate is taken to be the Eurodollar deposit rate and trading costs are estimated by the linear model described in Appendix C. Turnover-induced trading costs are 1% during the period 1929–1955, .5% during the period 1956–1970 and .1% during the period 1971–2012.[4] Summary statistics for the strategies are in Table 2.

Table 2: Historical Performance

Panel A: 1929-2012 Source: Bonds, Target: Stocks $r^b = r_f$ no trading costs	Arithmetic Excess Return	Geometric Excess Return	Arithmetic Excess Borrowing Return	Geometric Excess Borrowing Return	Average Leverage	Volatility	Sharpe Ratio	Skewness	Excess Kurtosis
Stocks	6.95	5.27	6.95	5.27	1.00	18.93	0.37	0.18	7.46
Bonds	1.59	1.55	1.59	1.55	1.00	3.26	0.49	0.03	4.74
FLT ($\lambda = 5$)	7.97	6.84	7.97	6.84	5.00	16.32	0.49	0.03	4.74
UVT ($\sigma = 19\%$)	8.49	6.17	8.49	6.17	8.75	22.11	0.38	-0.29	4.55
CVT	9.44	7.68	9.44	7.68	6.82	19.80	0.48	-0.51	6.44

Panel B: 1929-2012 Source: Bonds, Target: Stocks r^b = 3M-EDR with trading costs	Arithmetic Excess Return	Geometric Excess Return	Arithmetic Excess Borrowing Return	Geometric Excess Borrowing Return	Average Leverage	Volatility	Sharpe Ratio	Skewness	Excess Kurtosis
Stocks	6.95	5.27	6.17	4.46	1.00	18.93	0.37	0.18	7.46
Bonds	1.59	1.55	0.82	0.77	1.00	3.26	0.49	0.03	4.74
FLT ($\lambda = 5$)	4.22	2.92	3.44	2.14	5.00	16.34	0.26	-0.06	4.65
UVT ($\sigma = 19\%$)	-0.45	-3.02	-1.22	-3.76	8.75	22.55	-0.02	-0.58	4.81
CVT	2.61	0.57	1.84	-0.20	6.82	19.91	0.13	-0.68	6.71

Table 2: Summary statistics of the source and target portfolios and the levered strategies over the long sample period, December 1929–January 2012.

3.2.1 The Covariance Between Leverage and Borrowing Excess Return Can Substantially Affect the Cumulative Return to a Levered Strategy

The top panel of Figure 1 shows cumulative returns to FLT, UVT and CVT over our long sample in a friction-free market. The levered strategies were financed at the risk-free rate and trading is free. All three levered strategies earned more than the source portfolio. One dollar invested in the source portfolio in 1929 was worth \$68 in 2012. The corresponding

[4]This amounts to setting κ equal to .01, .005 or .001 in formula (22).

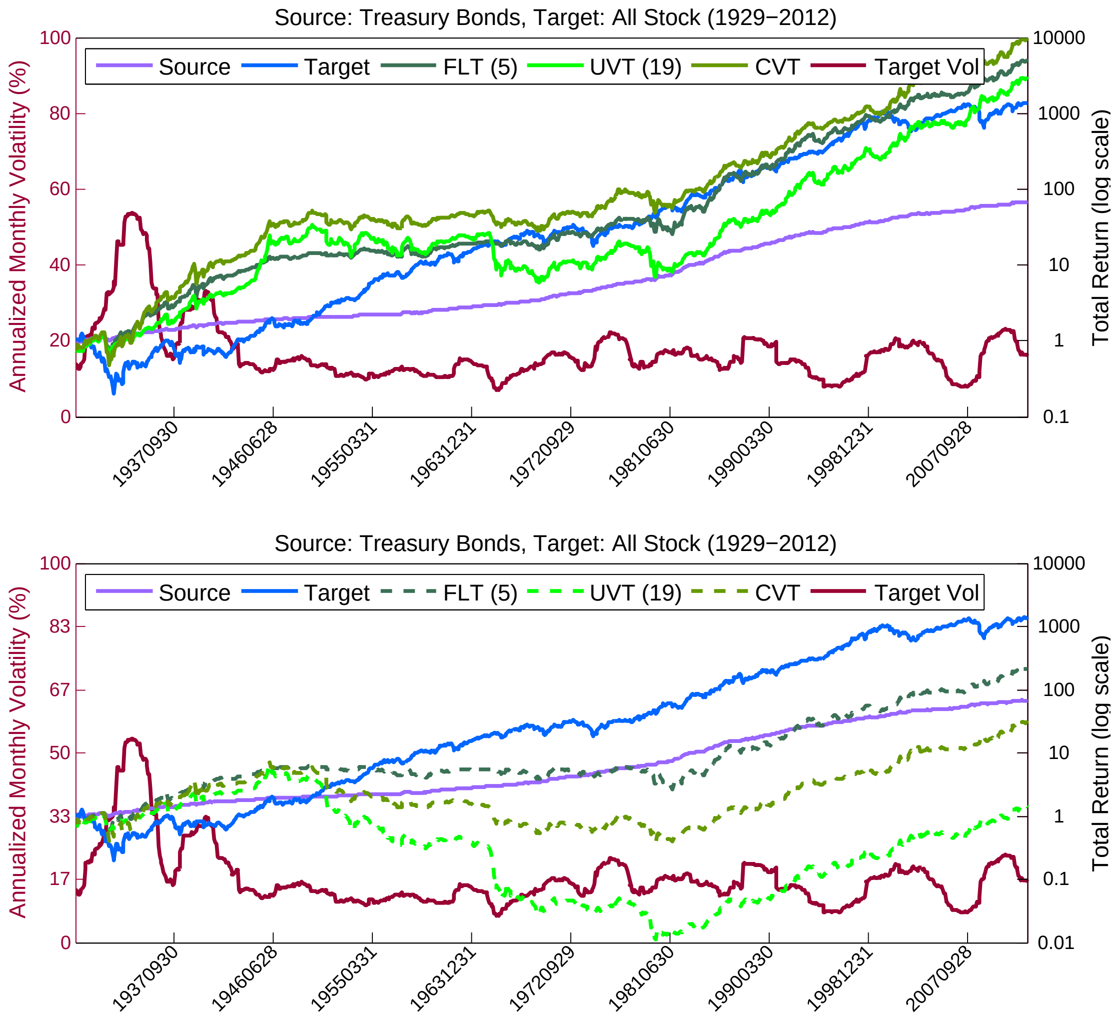

Figure 1: Cumulative returns to five investment strategies over the horizon January 1929 to December 2012; the CRSP value-weighted bond index (source), the CRSP value-weighted equity index (target), bonds 5-fold levered (FLT), bonds unconditionally levered to a 19% volatility target (UVT), and bonds conditionally levered to the volatility of equity (CVT). Also shown is the volatility of the conditional target. Top Panel: Leverage is financed at the T-Bill rate and there is no charge for turnover-induced trading. Bottom panel: Leverage is financed at the Eurodollar deposit rate and trading costs are based on the linear model in Appendix C.

amounts for the levered strategies are impressive: \$4822 for FLT, \$2841 for UVT and \$9301 and CVT. Note, however, that UVT returned less than the other levered strategies despite the fact that it had the highest leverage, as shown in Panel A of Table 3.

Table 3: Performance Attribution (Arithmetic Returns)

Panel A: 1929-2012 Source: Bonds. Target: Stocks $r^b = r_f$, no trading costs		FLT(5)		UVT (19)		CVT
$\mathbb{E}[\lambda - 1]$	4.00		7.75		5.82	
$\mathbb{E}[r^S - r^b]$	1.59		1.59		1.59	
$\mathbb{E}[\lambda - 1] \cdot \mathbb{E}[r^S - r^b]$		6.38		12.36		9.28
$\sigma(\lambda)$	0.00		20.51		12.25	
$\sigma(r^S - r^b)$	3.26		3.26		3.26	
$\rho(\lambda, r^S - r^b)$	0.00		-0.08		-0.04	
$\mathbf{C}ov(\lambda, r^S - r^b)$		0.00		-5.46		-1.44
$-\mathbb{E}[r^{TC}]$		0.00		0.00		0.00
$\mathbb{E}[r^L - r^S]$		6.38		6.89		7.84

Panel B: 1929-2012 Source: Bonds. Target: Stocks r^b = 3M-EDR, with trading costs		FLT(5)		UVT (19)		CVT
$\mathbb{E}[\lambda - 1]$	4.00		7.75		5.82	
$\mathbb{E}[r^S - r^b]$	0.82		0.82		0.82	
$\mathbb{E}[\lambda - 1] \cdot \mathbb{E}[r^S - r^b]$		3.29		6.37		4.79
$\sigma(\lambda)$	0.00		20.51		12.25	
$\sigma(r^S - r^b)$	3.27		3.27		3.27	
$\rho(\lambda, r^S - r^b)$	0.00		-0.07		-0.03	
$\mathbf{C}ov(\lambda, r^S - r^b)$		0.00		-4.98		-1.06
$-\mathbb{E}[r^{TC}]$		-0.67		-3.44		-2.71
$\mathbb{E}[r^L - r^S]$		2.62		-2.04		1.02

Table 3: Performance attribution of the realized arithmetic return of the levered strategies, FLT, UVT and CVT, in terms of their common source portfolio, a US treasury bond index, over the period January 1929–December 2012. The performance attribution is based on Formula (9). The results in Panel A come from a friction-free world in which borrowing is at the T-Bill rate and trading is free. The results in Panel B are adjusted for transaction costs; borrowing is at the Eurodollar deposit rate and trading costs are based on the linear model in Appendix C. Arithmetic returns are estimated from monthly data and annualized by multiplication by 12; they are displayed in percent.

Why did higher leverage translate into lower return? As shown in Formula (9), the covariance of leverage with borrowing excess return of the source contributes to the difference in expected return of a levered strategy and its source. Panel A of Table 3 shows that this contribution can be substantial: the covariance reduced realized return by 1.44% per year for CVT and 5.46% per year for UVT. Note that the covariance term negated almost half of the 12.36% annual return contributed by the product of excess borrowing return to the source and expected leverage minus one. Of course, the covariance term was zero for FLT by construction.

3.2.2 Transaction Costs Can Substantially Affect the Cumulative Return to a Levered Strategy

The study depicted in the top panel of Figure 1 was carried out in an idealized world where leverage can be financed at the risk-free rate and trading is free. The bottom panel of Figure 1 reports on an otherwise identical study that takes account of transaction costs due to leverage. Financing is at the Eurodollar deposit rate, and trading costs are linear in leverage-induced turnover. The close connection between leverage and leverage-induced turnover in the three strategies is illustrated in Figure 2.

Note in Figure 1 that after transaction costs, both UVT and CVT underperform the source strategy and the ranking of CVT and FLT is reversed. A dollar invested at the beginning of the 84-year long sample become \$30 for CVT, and \$1.41 for UVT. A dollar grew to \$209 for the fixed leverage strategy, FLT. These amounts are, to put it mildly, less impressive than the friction-free amounts. The results are explained, in part, by trading costs, which drained .67% from FLT, 2.71% from CVT and 3.44% from UVT in realized annual arithmetic return, as shown in Panel B of Table 3. Note, however that CVT appears to outperform the source portfolio, when arithmetic return is used. This conflicts with the ranking in Figure 1, a conflict that we will resolve shortly by considering geometric returns.

An additional consideration is that the higher borrowing costs are more damaging to strategies with higher leverage. An indication of the impact of realistic borrowing costs is found by comparing the product of excess borrowing return to the source and expected leverage minus one before and after transaction costs. The numbers are in Panels A and B of Table 3; realistic borrowing costs cut the contribution roughly in half for all three levered strategies.

3.2.3 Period Start and End Dates Can Substantially Affect the Cumulative Return to a Levered Strategy

A different perspective emerges in the last three decades, where UVT outperformed both FLT and CVT. Even after adjusting for transaction costs due to leverage, all three levered strategies outperformed the source portfolio. A dollar invested in the source portfolio in 1981 was worth \$9 in 2012, but it was worth \$70 in UVT, \$65 in CVT and \$58 in FLT. In this recent period, the leverage implied by the volatility targeting strategies was relatively low and relatively stable, as illustrated in the bottom panel of Figure 2. As a consequence, the average trading costs were relatively low as reported in Panel B of Table 4.

The era beginning in 1961 and ending in 1980 tells a strongly contrasting story. Even before adjusting for transaction costs, all three levered strategies returned less than the source portfolio. Leverage in the volatility targeting strategies was erratic and it reached unusually high levels in the mid-1960s. The erratic leverage translated into high transaction costs for the volatility targeting strategies, as shown in Table 5. This contributes materially to strategy performance. After accounting for leverage-related transaction costs,

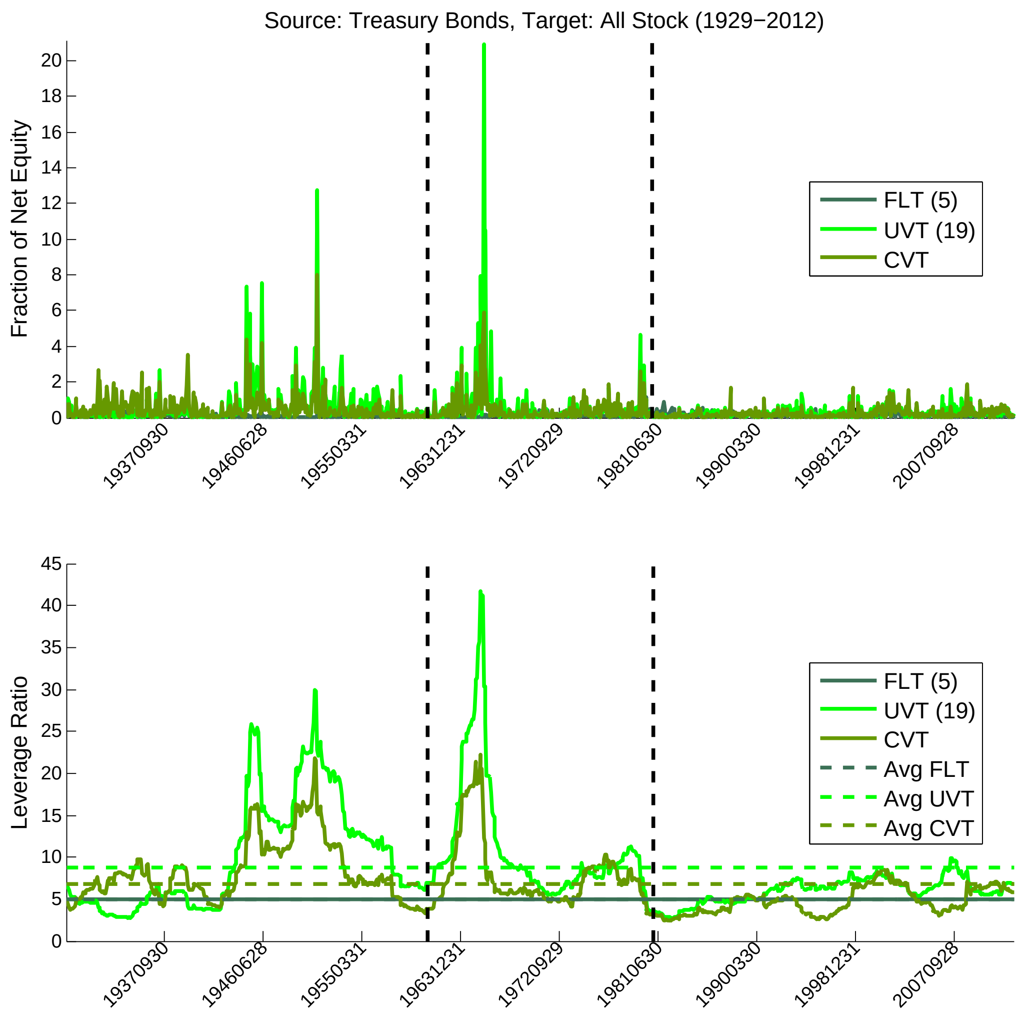

Figure 2: Top panel: Leverage induced turnover in FLT, UVT and CVT over the horizon January 1929 to December 2012 where the source is the the CRSP value-weighted bond index, the unconditional target is 19% per annum and the conditional target is volatility of the CRSP value-weighted equity index. Bottom panel: leverage implicit in FLT, UVT and CVT. The dashed vertical lines mark the subperiods, January 1960–December 1980 and January 1981–December 2012, discussed in Section 3.2.3.

Table 4: Performance Attribution (Arithmetic Returns)

Panel A: 1981-2012 Source: Bonds. Target: Stocks $r^b = r_f$, no trading costs		FLT(5)		UVT (19)		CVT
$\mathbb{E}[\lambda - 1]$	4.00		4.08		3.77	
$\mathbb{E}[r^S - r^b]$	2.51		2.51		2.51	
$\mathbb{E}[\lambda - 1] \cdot \mathbb{E}[r^S - r^b]$		10.03		10.24		9.45
$\sigma(\lambda)$	0.00		4.37		5.35	
$\sigma(r^S - r^b)$	3.31		3.31		3.31	
$\rho(\lambda, r^S - r^b)$	0.00		0.01		0.02	
$\mathbf{C}ov(\lambda, r^S - r^b)$		0.00		0.08		0.37
$-\mathbb{E}[r^{TC}]$		0.00		0.00		0.00
$\mathbb{E}[r^L - r^S]$		10.03		10.31		9.82

Panel B: 1981-2012 Source: Bonds. Target: Stocks r^b = 3M-EDR, with trading costs		FLT(5)		UVT (19)		CVT
$\mathbb{E}[\lambda - 1]$	4.00		4.08		3.77	
$\mathbb{E}[r^S - r^b]$	1.79		1.79		1.79	
$\mathbb{E}[\lambda - 1] \cdot \mathbb{E}[r^S - r^b]$		7.18		7.33		6.76
$\sigma(\lambda)$	0.00		4.37		5.35	
$\sigma(r^S - r^b)$	3.31		3.31		3.31	
$\rho(\lambda, r^S - r^b)$	0.00		0.03		0.04	
$\mathbf{C}ov(\lambda, r^S - r^b)$		0.00		0.40		0.73
$-\mathbb{E}[r^{TC}]$		-0.18		-0.23		-0.27
$\mathbb{E}[r^L - r^S]$		7.00		7.50		7.23

Table 4: Performance attribution of the realized arithmetic return of the levered strategies, FLT, UVT and CVT, in terms of their common source portfolio, a US treasury bond index, over the period January 1981–December 2012. The performance attribution is based on Formula (9). The results in Panel A come from a friction-free world in which borrowing is at the T-Bill rate and trading is free. The results in Panel B are adjusted for transaction costs; borrowing is at the Eurodollar deposit rate and trading costs are based on the linear model in Appendix C. Arithmetic returns are estimated from monthly data and annualized by multiplication by 12; they are displayed in percent.

the value in 1980 of a dollar invested in the source portfolio in 1961 was \$2.78; the analogous values for the levered strategies are \$.70 for FLT, \$.27 for CVT and only \$.11 for UVT! Note that these values are not adjusted for inflation, which was particularly high in the latter part of the period. By comparing Panel B of Table 5 with Panel B of Table 4, we see how different trading costs can be from one period to another. For example trading costs for UVT led to a drag on return of 0.23% per year between 1981 and 2012, but they were ten times greater at 2.32% per year between 1961 and 1980. FLT benefited from lower leverage than UVT and CVT in a period when the source portfolio had negative excess borrowing return, and from lower trading costs than UVT and CVT.

Table 5 also shows that leverage covaried negatively with excess borrowing return to the source for UVT and positively for CVT, both before and after transaction costs. This helps explain why CVT returns were slightly less disastrous than UVT returns over this period.

Table 5: Performance Attribution (Arithmetic Returns)

Panel A: 1961-1980 Source: Bonds. Target: Stocks $r^b = r_f$, no trading costs		FLT(5)		UVT (19)		CVT
$\mathbb{E}[\lambda - 1]$	4.00		7.73		6.81	
$\mathbb{E}[r^S - r^b]$	-0.25		-0.25		-0.25	
$\mathbb{E}[\lambda - 1] \cdot \mathbb{E}[r^S - r^b]$		-0.99		-1.91		-1.68
$\sigma(\lambda)$	0.00		19.96		14.20	
$\sigma(r^S - r^b)$	2.93		2.93		2.93	
$\rho(\lambda, r^S - r^b)$	0.00		-0.04		0.00	
$\mathbf{C}ov(\lambda, r^S - r^b)$		0.00		-2.56		0.20
$-\mathbb{E}[r^{TC}]$		0.00		0.00		0.00
$\mathbb{E}[r^L - r^S]$		-0.99		-4.46		-1.48

Panel B: 1961-1980 Source: Bonds. Target: Stocks r^b = 3M-EDR, with trading costs		FLT(5)		UVT (19)		CVT
$\mathbb{E}[\lambda - 1]$	4.00		7.73		6.81	
$\mathbb{E}[r^S - r^b]$	-1.40		-1.40		-1.40	
$\mathbb{E}[\lambda - 1] \cdot \mathbb{E}[r^S - r^b]$		-5.61		-10.84		-9.55
$\sigma(\lambda)$	0.00		19.96		14.20	
$\sigma(r^S - r^b)$	2.93		2.93		2.93	
$\rho(\lambda, r^S - r^b)$	0.00		-0.01		0.03	
$\mathbf{C}ov(\lambda, r^S - r^b)$		0.00		-0.81		1.42
$-\mathbb{E}[r^{TC}]$		-0.32		-2.32		-1.80
$\mathbb{E}[r^L - r^S]$		-5.93		-13.97		-9.94

Table 5: Performance attribution of the realized arithmetic return of the levered strategies, FLT, UVT and CVT, in terms of their common source portfolio, a US treasury bond index, over the period January 1961–December 1980. The performance attribution is based on Formula (9). The results in Panel A come from a friction-free world in which borrowing is at the T-Bill rate and trading is free. The results in Panel B are adjusted for transaction costs; borrowing is at the Eurodollar deposit rate and trading costs are based on the linear model in Appendix C. Arithmetic returns are estimated from monthly data and annualized by multiplication by 12; they are displayed in percent.

3.2.4 Compounding Can Substantially Affect the Cumulative Return to a Levered Strategy

The impact of compounding can be seen over the long sample in a friction-free market. If we rank the levered strategies by their average realized arithmetic excess return over their common source portfolio, as in Table 3, UVT outperforms FLT. However, as shown in Figure 1, a dollar invested in FLT in 1929 is worth more in 2012 than a dollar in UVT. This reversal arises because average realized arithmetic returns do not properly account for compounding.

Therefore, we turn to Formula (11), which expresses the difference between the log geometric returns of the levered stratey and the source, $\log\left(1 + G\left(r^{\mathbf{L}}\right)\right) - \log\left(1 + G\left(r^{\mathbf{S}}\right)\right)$, in terms of the diffrence in arithmetic returns, $E\left[r^{\mathbf{L}} - r^{\mathbf{S}}\right]$, and a compounding correction involving the expected returns and variances of the levered strategy and the source portfolio. Table 6, which shows that the adjustment for variance was much greater for UVT, before and after transaction costs, than for the other strategies.

Breakdowns of geometric returns for our two shorter periods, January 1981–December 2012 and January 1961–December 1980, are in Tables 7 and 8.

Table 6: Performance Attribution (Geometric Returns)

Panel A: 1929-2012 Source: Bonds. Target: Stocks $r^b = r_f$, no trading costs		FLT(5)		UVT (19)		CVT
$\mathbb{E}[r^L - r^S]$	6.38		6.89		7.84	
$1 - 0.5(\mathbb{E}[r^L] + \mathbb{E}[r^S])$	0.99		0.99		0.99	
$\mathbb{E}[r^L - r^s] \cdot \{1 - 0.5(\mathbb{E}[r^L] + \mathbb{E}[r^S])\}$		6.33		6.84		7.78
$\mathbf{V}ar(r^L)$	2.65		4.86		3.89	
$\mathbf{V}ar(r^S)$	0.11		0.11		0.11	
$-0.5[\mathbf{V}ar(r^L) - \mathbf{V}ar(r^S)]$		-1.27		-2.37		-1.89
$\log(1 + \mathbb{G}[r^L]) - \log(1 + \mathbb{G}[r^S])$		5.08		4.45		5.86

Panel B: 1929-2012 Source: Bonds. Target: Stocks r^b = 3M-EDR, with trading costs		FLT(5)		UVT (19)		CVT
$\mathbb{E}[r^L - r^S]$	2.62		-2.04		1.02	
$1 - 0.5(\mathbb{E}[r^L] + \mathbb{E}[r^S])$	0.99		1.00		1.00	
$\mathbb{E}[r^L - r^s] \cdot \{1 - 0.5(\mathbb{E}[r^L] + \mathbb{E}[r^S])\}$		2.61		-2.03		1.01
$\mathbf{V}ar(r^L)$	2.65		5.06		3.94	
$\mathbf{V}ar(r^S)$	0.11		0.11		0.11	
$-0.5[\mathbf{V}ar(r^L) - \mathbf{V}ar(r^S)]$		-1.27		-2.47		-1.91
$\log(1 + \mathbb{G}[r^L]) - \log(1 + \mathbb{G}[r^S])$		1.34		-4.61		-0.97

Table 6: Performance attribution of the realized geometric return of the levered strategies, FLT, UVT and CVT, in terms of their common source portfolio, a US treasury bond index, over the period January 1929–December 2012. The performance attribution is based on Formula (11). The results in Panel A come from a friction-free world in which borrowing is at the T-Bill rate and trading is free. The results in Panel B are adjusted for transaction costs; borrowing is at the Eurodollar deposit rate and trading costs are based on the linear model in Appendix C. Geometric returns are estimated from monthly data. The panel shows log geometric returns $\log(1 + G(r))$, annualized by multiplication by 12; they are displayed in percent.

4 The Decision to Lever

The performance attribution scheme developed in Section 2 provides a framework for choosing a fully-invested source portfolio, and determining whether and how to lever it:

(1) Choose a source portfolio, $\mathbf{S}$, with a high forecast Sharpe ratio:

$$\frac{E(\mathbf{S}) - r^f}{\sigma(\mathbf{S})}, \tag{12}$$

over a given investment horizon.

Table 7: Performance Attribution (Geometric Returns)

Panel A: 1981-2012 Source: Bonds. Target: Stocks $r^b = r_f$, no trading costs		FLT(5)		UVT (19)		CVT
$\mathbb{E}[r^L - r^S]$	10.03		10.31		9.82	
$1 - 0.5(\mathbb{E}[r^L] + \mathbb{E}[r^S])$	0.99		0.99		0.99	
$\mathbb{E}[r^L - r^s] \cdot \{1 - 0.5(\mathbb{E}[r^L] + \mathbb{E}[r^S])\}$		9.93		10.21		9.72
$\mathbf{V}ar(r^L)$	2.74		2.54		2.46	
$\mathbf{V}ar(r^S)$	0.12		0.12		0.12	
$-0.5[\mathbf{V}ar(r^L) - \mathbf{V}ar(r^S)]$		-1.31		-1.21		-1.17
$\log(1 + \mathbb{G}[r^L]) - \log(1 + \mathbb{G}[r^S])$		8.66		9.04		8.58

Panel B: 1981-2012 Source: Bonds. Target: Stocks r^b = 3M-EDR, with trading costs		FLT(5)		UVT (19)		CVT
$\mathbb{E}[r^L - r^S]$	7.00		7.50		7.23	
$1 - 0.5(\mathbb{E}[r^L] + \mathbb{E}[r^S])$	0.99		0.99		0.99	
$\mathbb{E}[r^L - r^s] \cdot \{1 - 0.5(\mathbb{E}[r^L] + \mathbb{E}[r^S])\}$		6.94		7.43		7.16
$\mathbf{V}ar(r^L)$	2.72		2.52		2.45	
$\mathbf{V}ar(r^S)$	0.12		0.12		0.12	
$-0.5[\mathbf{V}ar(r^L) - \mathbf{V}ar(r^S)]$		-1.30		-1.20		-1.17
$\log(1 + \mathbb{G}[r^L]) - \log(1 + \mathbb{G}[r^S])$		5.67		6.26		6.02

Table 7: Performance attribution of the realized geometric return of the levered strategies, FLT, UVT and CVT, in terms of their common source portfolio, a US treasury bond index, over the period January 1981–December 2012. The performance attribution is based on Formula (11). The results in Panel A come from a friction-free world in which borrowing is at the T-Bill rate and trading is free. The results in Panel B are adjusted for transaction costs; borrowing is at the Eurodollar deposit rate and trading costs are based on the linear model in Appendix C. Geometric returns are estimated from monthly data. The panel shows log geometric returns $\log(1 + G(r))$, annualized by multiplication by 12; they are displayed in percent.

(a) Among candidate source portfolios with roughly equal forecast Sharpe ratios, choose a portfolio with relatively high volatility, since the desired mean return and volatility can be achieved with lower leverage.

(2) Estimate the borrowing Sharpe ratio:

$$\frac{E(\mathbf{S}) - r^b}{\sigma(\mathbf{S})}, \tag{13}$$

for this portfolio. Assuming that the risk-free and borrowing rates are constant, the forecast borrowing Sharpe ratio of $\mathbf{S}$ will also be high. However, unlike the Sharpe ratio, the borrowing Sharpe ratio is invariant to the level of leverage, provided the level of leverage is held constant. Thus, the borrowing Sharpe ratio provides a better picture of the potential returns to the levered strategy than does the regular Sharpe ratio.

Table 8: Performance Attribution (Geometric Returns)

Panel A: 1961-1980 Source: Bonds. Target: Stocks $r^b = r_f$, no trading costs		FLT(5)		UVT (19)		CVT
$\mathbb{E}[r^L - r^S]$	-0.99		-4.46		-1.48	
$1 - 0.5(\mathbb{E}[r^L] + \mathbb{E}[r^S])$	1.00		1.00		1.00	
$\mathbb{E}[r^L - r^s] \cdot \{1 - 0.5(\mathbb{E}[r^L] + \mathbb{E}[r^S])\}$		-0.98		-4.45		-1.47
$\mathbf{V}ar(r^L)$	2.12		4.08		3.25	
$\mathbf{V}ar(r^S)$	0.08		0.08		0.08	
$-0.5[\mathbf{V}ar(r^L) - \mathbf{V}ar(r^S)]$		-1.02		-2.00		-1.58
$\log(1 + \mathbb{G}[r^L]) - \log(1 + \mathbb{G}[r^S])$		-1.97		-6.51		-3.05

Panel B: 1961-1980 Source: Bonds. Target: Stocks r^b = 3M-EDR, with trading costs		FLT(5)		UVT (19)		CVT
$\mathbb{E}[r^L - r^S]$	-5.93		-13.97		-9.94	
$1 - 0.5(\mathbb{E}[r^L] + \mathbb{E}[r^S])$	1.00		1.00		1.00	
$\mathbb{E}[r^L - r^s] \cdot \{1 - 0.5(\mathbb{E}[r^L] + \mathbb{E}[r^S])\}$		-5.92		-14.00		-9.93
$\mathbf{V}ar(r^L)$	2.11		4.24		3.31	
$\mathbf{V}ar(r^S)$	0.08		0.08		0.08	
$-0.5[\mathbf{V}ar(r^L) - \mathbf{V}ar(r^S)]$		-1.01		-2.08		-1.61
$\log(1 + \mathbb{G}[r^L]) - \log(1 + \mathbb{G}[r^S])$		-6.93		-16.20		-11.58

Table 8: Performance attribution of the realized geometric return of the levered strategies, FLT, UVT and CVT, in terms of their common source portfolio, a US treasury bond index, over the period January 1961–December 1980. The performance attribution is based on Formula (11). The results in Panel A come from a friction-free world in which borrowing is at the T-Bill rate and trading is free. The results in Panel B are adjusted for transaction costs; borrowing is at the Eurodollar deposit rate and trading costs are based on the linear model in Appendix C. Geometric returns are estimated from monthly data. The panel shows log geometric returns $\log(1 + G(r))$, annualized by multiplication by 12; they are displayed in percent.

(3) Choose a leverage rule:

   (a) Constant leverage is a reasonable default.

   (b) If there is reason to believe that a particular leverage rule will produce a positive correlation,

      i. Estimate the trading costs. Does the extra return from the positive correlation transcend the trading costs? If not, go back to constant leverage.

      ii. Estimate the compounding correction. Does the extra return from a positive correlation survive after trading costs and the compounding correction? If not, go back to constant leverage.

(4) Choose the scale of leverage. Any leverage rule specifies a leverage profile that depends on future market conditions. You can raise or lower the profile by multiplying leverage by a scalar γ. Choose γ, keeping in mind:

(a) Increasing γ may increase expected return, but it also increases the volatility of the strategy as well as other risks.

(b) Increasing γ increases the impact of trading costs and the compounding correction. At each instant in time

i. The borrowing cost increases linearly in γ.

ii. Under our assumption of linear trading costs, which ignores market impact, the trading cost increases roughly linearly in γ. However, in a model taking into account market impact, the trading cost would likely increase quadratically in γ.

iii. The compounding correction increases quadratically in γ.

(c) Increasing γ increases the risk of a forced liquidation of a levered strategy at an inopportune moment. An example is the March 2008 collapse of Carlyle Capital Corporation, whose source portfolio was Fannie Mae and Freddie Mac guaranteed Mortgage Backed Securities, levered approximately 30 to 1; see Werdiger (2008). This is a substantial risk that is beyond the scope of this paper.

5 Setting Targets

There is no ideal way to create equivalent UVT and CVT strategies. In this study, we set the unconditional volatility target to the realized volatility of US equity which was 19% between January 1929 and December 2012, 16% between January 1981 and December 2012, and 15% between January 1961 and December 1980. An advantage of this method is that volatility-adjusted return is easy to discern. Disadvantages of this method include lookahead bias and disregard of risks beyond volatility.

In all of our studies, leverage in FLT is set to 5. This is approximately equal to average leverage in UVT and CVT in the most recent period. However, it is below average over the long sample, and in the period from between January 1961 and December 1980.

6 Conclusion

The potential for leverage to enhance the return to a portfolio is seductive, but there is little to recommend a levered strategy that does not outperform its underlying source portfolio. Looking backward, therefore, a rational assessment of a levered strategy is taken relative to its source portfolio. Similarly, a rational decision to lever depends on the expectation that a levered strategy will outperform its source portfolio over the investment horizon in question.

In this article, we develop a platform that supports both backward-looking performance attribution and forward-looking investment decisions concerning levered strategies.

Specifically, in Formula (9), we express the difference between arithmetic expected return to a levered strategy portfolio and its source as a sum of three terms:

$$E\left[r^{\mathbf{L}} - r^{\mathbf{S}}\right] = E\left[\lambda - 1\right] E\left[r^{\mathbf{S}} - r^{b}\right] + \mathrm{cov}(\lambda, r^{\mathbf{S}} - r^{b}) - E\left[r^{\mathbf{TC}}\right].$$

The first term is the one that most easily comes to mind in the context of a levered strategy; it is the product of average leverage minus one and expected borrowing excess return of the source portfolio. If this term were the only consideration, then more leverage would lead to higher expected return as long as the expected return to the source portfolio is greater than the expected borrowing rate. However, as we have shown empirically in three strategies that lever US treasury bonds, other factors have a material effect on the cumulative return to a levered strategy. These are: covariance of leverage with the excess borrowing return, realistic financing costs, trading costs and compounding effects. Our studies demonstrate that any of these terms can reverse the ranking of strategies.

Formula (9) accounts for both the covariance term and transaction costs. However, it neglects the effect of compounding, which imposes a drag on cumulative return that is not captured in arithmetic expected return. If the levered strategy has high or unstable volatility, the drag may be substantial. Hence a more complete attribution and a more accurate decision rule depends on geometric expected return in Formula (11):

$$\begin{aligned} &\log\left(1 + G\left[r^{\mathbf{L}}\right]\right) - \log\left(1 + G\left[r^{\mathbf{S}}\right]\right) \\ \sim \quad & E\left[r^{\mathbf{L}} - r^{\mathbf{S}}\right]\left(1 - \frac{E\left[r^{\mathbf{L}}\right] + E\left[r^{\mathbf{S}}\right]}{2}\right) - \frac{\mathrm{var}\left(r^{\mathbf{L}}\right) - \mathrm{var}\left(r^{\mathbf{S}}\right)}{2} \end{aligned}$$

As we have shown, the volatility term,

$$\frac{\mathrm{var}\left(r^{\mathbf{L}}\right) - \mathrm{var}\left(r^{\mathbf{S}}\right)}{2},$$

can be substantial enough to reverse arithmetic-rule-based rankings of strategies.

In this paper, we examine the realized performance of three levered strategies: fixed leverage targeting (FLT), unconditional volatility targeting (UVT) and conditional volatility targeting (CVT). Some scholars have expressed the view that CVT is a poor strategy compared to UVT.[5] By contrast, we find that FLT, UVT and CVT are different from one another in important ways, and they perform very differently under different market conditions. In fact, it is the leverage that is implicitly determined by the volatility targets in UVT and CVT, and not the volatility itself, that interacts with the return to the source portfolio to determine strategy performance.

The period following the financial crisis has featured Fed-supported interest rates that are extraordinary low by historical standards. As quantitative easing comes to an end, the cost of funding a levered strategy will rise dramatically, and historical precedent suggests

[5]See Asness et al. (2013).

that the impact may well be amplified by declines in asset prices. These considerations should be incorporated in any decision to lever.

Appendices

A Related Literature

A.1 CAPM

Finance continues to draw heavily on the Capital Asset Pricing Model (CAPM) developed in Treynor (1961), Treynor (1962), Sharpe (1964), Lintner (1965b), Lintner (1965a), Mossin (1966), and extended in Black and Litterman (1992).[6] Here, leverage is a means to adjust the level of risk in an efficient portfolio and nothing more. In contrast, Markowitz (2005) illustrates another facet of leverage in the context of a market composed of three coconut farms. In this disarmingly simple example, some investors are leverage-constrained and others are not. The market portfolio is mean-variance inefficient; as a result, no mean-variance investor chooses to hold it, and expected returns of assets do not depend linearly on market betas.

A.2 Measurement of Risk and Nonlinearities

An impediment to a clear understanding of leverage may be the way we measure its risk. Standard risk measures such as volatility, value at risk, expected shortfall, and beta scale linearly with leverage. But as we know from the collapse of Long Term Capital, the relationship between risk and leverage can be non-linear; see, for example, Jorion (2000). Föllmer and Schied (2002) and Föllmer and Schied (2011, Chapter 4) describe risk measures that penalize leverage in a super-linear way. Recent experience suggests that these measures may be useful in assessing the risk of levered strategies.

One contribution of this paper is to explain how the interaction between leverage and market frictions creates specific nonlinearities in the relationship between leverage and return. Understanding these specific nonlinearities provides a practical framework to guide the decision on whether and how to leverage.

A.3 Motivations for Leverage

If investors are overconfident in their predictions of investment returns, they may find leverage attractive because it magnifies the returns when times are good, and because

[6] A history of the CAPM elucidating Jack Treynor's role in its development is in French (2003).

they underestimate the risk of bad outcomes.[7]

Perfectly rational investors may also be attracted to leverage by the low risk anomaly, the apparent tendency of certain low-risk portfolios to have higher risk-adjusted return than high-risk portfolios. An investor who believes in the low risk anomaly will be tempted to lever low-risk portfolios, in the hope of achieving high expected returns at acceptable levels of risk.

In a CAPM world, investors with below-average risk aversion will choose to lever the market portfolio.[8] The low risk anomaly provides a rational argument for investors with typical risk aversion to use leverage. Indeed, the low risk anomaly is arguably the only rational argument for an investor to use leverage in an investment portfolio composed of publicly traded securities.[9] Differences in risk aversion could explain some investors choosing higher expected return at the price of higher volatility, but there is little reason for a rational investor to choose *leverage* unless the source portfolio being levered offers superior *risk-adjusted* returns, at a volatility *below* the investor's risk tolerance.

A.4 Levered Low-Risk Strategies

Low-risk investing refers to a diverse collection of investment strategies that emphasize low beta, low idiosyncratic risk, low volatility or downside protection. The collection of low-risk strategies includes broad asset allocations, but it also includes narrower strategies restricted to a single asset class. An early reference to low-risk investing is Markowitz (1952) who comments that a minimum-variance portfolio is mean-variance optimal if all assets returns are uncorrelated and have equal expectations. But low-risk strategies typically require leverage in order to meet expected return targets. In an exploration of this idea, Frazzini and Pedersen (2010) echo some of the conclusions in Markowitz (2005), and they complement theory with an empirical study of an implicitly levered equity risk factor that is long low-beta stocks and short high-beta stocks. This factor descends from Black et al. (1972), which provides evidence that the (CAPM) may not properly reflect market behavior.

[7] A positive relationship between overconfident CEOs and firm leverage is documented in Malmendier et al. (2011). Shefrin and Statman (2011) identify excessive leverage taken by overconfident bankers as a contributor to the global financial crisis.

[8] Note, however, that the market portfolio in CAPM includes bonds and other risky asset classes, rather than just stocks. Levered strategies include the use of margin, and futures and other derivatives, to assemble levered equity-only portfolios, which behave quite differently from levered portfolios in CAPM.

[9] There are, of course, other rational arguments for using leverage in other contexts. The leverage provided by a mortgage may be the only feasible way for a household to buy a house, which provides a stream of consumption benefits and tax advantages in addition to facilitating an investment in the real estate market. Companies leverage their shareholder equity with borrowing to finance operations, for a variety of reasons, including differences in risk aversion, informational asymmetries, and tax implications.

A.5 Empirical Evidence on Levered Low-Risk Investing

There is a growing empirical literature indicating that market frictions may present investors from harvesting the returns promised by a frictionless analysis of levered low-risk strategies. Anderson et al. (2012) show that financing and trading costs can negate the abnormal profits earned by a levered risk parity strategy in a friction-free market. Li et al. (2013) and Fu (2009) show that market frictions may impede the ability to scale up the return of low-risk strategies through leverage.[10]

Asset allocation that is based on capital weights has a long and distinguished history; see, for example Graham (1949) and Bogle (2007). However, rules-based strategies that allocate risk instead of, or in addition to, capital are of a more recent vintage. Risk-based investing is discussed in Lörtscher (1990), Kessler and Schwarz (1996), Qian (2005), Clarke et al. (2011), Shah (2011), Sefton et al. (2011), Clarke et al. (2013), Anderson et al. (2012), Cowan and Wilderman (2011), Bailey and de Prado (2012), Goldberg and Mahmoud (2013) and elsewhere. Strategies that target volatility are also gaining acceptance, although the literature is still sparse. Goldsticker (2012) compares volatility targeting strategies to standard allocations such as fixed mix, and finds that the relative performance of the strategies is period dependent.

A.6 The Effect of Leverage on Markets

Another important question is the extent to which leverage may *contribute* to market instability. See, for example, Brunnermeier and Pedersen (2009), Adrian and Shin (2010) and Geanakoplos (2010). We do not address that question here, as we restrict our analysis to the effect of leverage on the return of investment strategies, taking the distribution of the underlying asset returns as given.

B Data

The results presented in this paper are based on CRSP stock and bond data from January of 1929 through December of 2012. The aggregate stock return is the CRSP value weighted market return (including dividends) from the table *Monthly Stock–Market Indices (NYSE/AMEX/NASDAQ)* – variable name *vwretd*. The aggregate bond return is the face value outstanding (cross-sectionally) weighted average of the unadjusted return for each bond in the *CRSP Monthly Treasury (Master)* table. In this table, the variable name for the unadjusted return is *retnua* and for the face value outstanding is *iout1r*. All bonds in the table are used, provided the values for both *retnua* and *iout1r* are not missing.

[10]Ross (2004) provides an example of the limits to arbitraging mispricings of interest-only strips of mortgage backed securities.

The proxy for the risk-free rate is the *USA Government 90-day T-Bills Secondary Market* rate, provided by Global Financial Data (http://www.globalfinancialdata.com), covering the period from January of 1929 through December of 2012. The proxy for the cost of financing leverage is the *U.S. 3-Month Euro-Dollar Deposit* rate, downloaded from the Federal Reserve (http://www.federalreserve.gov/releases/h15/data.htm). The 3-Month Euro-Dollar Deposit data is available from January of 1971 through December of 2012. Prior to January of 1971, a constant of 60 basis points is added to the 90-day T-Bill rate.[11]

C Trading Costs

We estimate the drag on return that stems from the turnover-induced trading required to maintain leverage targets in a strategy that levers a source portfolio S.

At time t, the strategy calls for an investment with a leverage ratio of λ_t. We make the harmless assumption that the value of the levered strategy at t, denoted L_t is \$1.[12] Then the holdings in the source portfolio, or assets, are $A_t = \lambda_t$. The debt, or liability, at time t is given by $D_t = \lambda_t - 1$.

Between times t and $t+1$, the value of the source portfolio changes from S_t to S_{t+1} and the strategy calls for rebalancing to achieve leverage λ_{t+1}. Just prior to rebalancing, the value of the investment is

$$A'_t = \lambda_t(1 + r_t^{\mathbf{S}}), \tag{14}$$

the liability has grown to $D'_t = (\lambda_t - 1)(1 + r_t^b)$ and the investor's equity is:

$$\begin{aligned} L'_t &= A'_t - D'_t \\ &= \lambda_t(1 + r_t^{\mathbf{S}}) - (\lambda_t - 1)(1 + r_t^b). \end{aligned} \tag{15}$$

At time $t+1$, the strategy is rebalanced according to its rules, and we let x_t denote the dollar amount of the change in value due to rebalancing, so that:

$$x_t = A_{t+1} - A'_t. \tag{16}$$

If we assume a linear model, the cost of trading x_t is $\kappa|x_t|$ for some $\kappa > 0$. The cost reduces the investor's equity to:

$$\begin{aligned} L_{t+1} &= L'_t - \kappa|x_t| \\ &= \lambda_t(1 + r_t^{\mathbf{S}}) - (\lambda_t - 1)(1 + r_t^b) - \kappa|x_t| \end{aligned} \tag{17}$$

[11] The average difference between the 90-day T-Bill Rate and the 3-Month Euro-Dollar Deposit Rate from 1971 through 2012 is 102 basis points. So our estimate of 60 basis points is relatively conservative.

[12] This assumption is harmless in a linear model of trading costs, which we develop here. It would be inappropriate for a realistic model of market impact.

in view of Formula (15), and the strategy rules combined with Formula (17) imply the value of the levered strategy at time $t+1$ must be:

$$\begin{aligned} A_{t+1} &= \lambda_{t+1} L_{t+1} \\ &= \lambda_{t+1} \left(\lambda_t (1 + r_t^{\mathbf{S}}) - (\lambda_t - 1)(1 + r_t^b) - \kappa |x_t| \right). \end{aligned} \tag{18}$$

Formulas (16), (18) and (14) imply:

$$x_t = \lambda_{t+1} \left(\left(\lambda_t (1 + r_t^{\mathbf{S}}) - (\lambda_t - 1)(1 + r_t^b) \right) - \kappa |x_t| \right) - \lambda_t (1 + r_t^S). \tag{19}$$

In the case that $\kappa = 0$, the solution is:

$$x_t^0 = \left((\lambda_{t+1} - 1) \lambda_t (1 + r_t^S) - (\lambda_t - 1) \lambda_{t+1} (1 + r_t^b) \right)$$

The solution for general κ is

$$x_t = \begin{cases} \frac{x_t^0}{1 + \kappa \lambda_{t+1}} & \text{if } x_t^0 \geq 0 \\ \\ \frac{x_0^t}{1 - \kappa \lambda_{t+1}} & \text{if } x_t^0 < 0 \end{cases} \tag{20}$$

provided that $\kappa \lambda_{t+1} < 1$.

The reduction in return due to trading costs, $r^{\mathbf{TC}}$, corresponding to Formula (20) is given by:

$$r_t^{\mathbf{TC}} = \begin{cases} \frac{\kappa x_t^0}{1 + \kappa \lambda_{t+1}} & \text{if } x_t^0 \geq 0 \\ \\ \frac{\kappa x_0^t}{1 - \kappa \lambda_{t+1}} & \text{if } x_t^0 < 0 \end{cases} \tag{21}$$

Even though the return estimate depends on current- and next-period leverage, it is possible to formulate $r^{\mathbf{TC}}$ without subscripts. Let F denote an operator that takes maps leverage in one period to leverage in the next. Then:

$$x^0 = \left((F(\lambda) - 1) \lambda (1 + r^S) - (\lambda - 1) F(\lambda) (1 + r^b) \right)$$

and

$$r^{\mathbf{TC}} = \begin{cases} \frac{\kappa x^0}{1 + \kappa F(\lambda)} & \text{if } x^0 \geq 0 \\ \\ \frac{\kappa x_0}{1 - \kappa F(\lambda)} & \text{if } x^0 < 0 \end{cases} \tag{22}$$

D Geometric Return

Let L_t denote the equity in a strategy at month t, where $t = 0, 1, \ldots, T$.

The correct ranking of realized strategy performance, taking compounding into account, is given by $G(r)$, the geometric average of the monthly returns, minus one:

$$\begin{aligned} G(r) &= \left(\frac{L_T}{L_0}\right)^{1/T} - 1 \\ &= \left[\prod_{t=0}^{T-1} \frac{L_{t+1}}{L_t}\right]^{1/T} - 1 \\ &= \left[\prod_{t=0}^{T-1} (1 + r_t)\right]^{1/T} - 1 \qquad (23) \end{aligned}$$

As we shall see, $\log(1 + G((r))$ meshes very nicely with our formulas for arithmetic return. Because the logarithm is strictly increasing, $G(r)$ and $\log(1 + G(r))$ induce exactly the same ranking of realized strategy returns.

$$\begin{aligned} \log(1 + G(r)) &= \frac{1}{T}\sum_{t=0}^{T-1} \log(1 + r_t) \\ &\sim \frac{1}{T}\sum_{t=0}^{T-1} \left(r_t - \frac{(r_t)^2}{2}\right) \qquad (24) \\ &= \frac{1}{T}\sum_{t=0}^{T-1} r_t - \frac{1}{T}\sum_{t=0}^{T-1} \frac{(r_t)^2}{2} \qquad (25) \end{aligned}$$

Formula (24) approximates $\log(1 + r_t)$ by its quadratic Taylor polynomial.[13]

Thus, we have:

$$\begin{aligned} &\log\left(1 + G\left[r^{\mathbf{L}}\right]\right) - \log\left(1 + G\left[r^{\mathbf{S}}\right]\right] \\ &= \frac{1}{T}\sum_{t=0}^{T-1} \left(r_t^{\mathbf{L}} - r_t^{S}\right) - \frac{1}{T}\sum_{t=0}^{T-1} \frac{\left(r_t^{\mathbf{L}}\right)^2 - \left(r_t^{\mathbf{S}}\right)^2}{2} \\ &= \frac{1}{T}\sum_{t=0}^{T-1} \left(r_t^{\mathbf{L}} - r_t^{S}\right)\left(1 - \frac{r_t^{\mathbf{L}} + r_t^{\mathbf{S}}}{2}\right) \\ &\sim E\left(r^{\mathbf{L}} - r^{\mathbf{S}}\right)\left(1 - \frac{E\left(r^{\mathbf{L}}\right) + E\left(r^{\mathbf{S}}\right)}{2}\right) - \frac{\operatorname{cov}\left(r^{\mathbf{L}} - r^{\mathbf{S}}, r^{\mathbf{L}} + r^{\mathbf{S}}\right)}{2} \\ &= E\left(r^{\mathbf{L}} - r^{\mathbf{S}}\right)\left(1 - \frac{E\left(r^{\mathbf{L}}\right) + E\left(r^{\mathbf{S}}\right)}{2}\right) - \frac{\operatorname{var}\left(r^{\mathbf{L}}\right) - \operatorname{var}\left(r^{\mathbf{S}}\right)}{2} \qquad (26) \end{aligned}$$

[13] Because the Taylor series for logarithm is alternating and decreasing in absolute value for $|r_t| < 1$, the error in the approximation of $\log(1 + r_t)$ is bounded above by $|r^t|^3/3$ for each month t. Since the monthly returns are both positive and negative, the errors in months with negative returns will substantially offset the errors in months with positive returns, so the errors will tend not to accumulate over time, in contrast with the second order terms which are all positive.

References

Tobias Adrian and Hyun Song Shin. Liquidity and Leverage. *J. Finan. Intermediation*, 19:418–437, 2010.

Robert M. Anderson, Stephen W. Bianchi and Lisa R. Goldberg. Will My Risk Parity Strategy Outperform? *Financial Analysts Journal*, 68(6):75–93, 2012.

Clifford Asness, Andrea Frazzini and Lasse Heje Pedersen. Will My Risk Parity Strategy Outperform?: A Comment. *Financial Analysts Journal*, 69(2):12–15, 2013.

David H. Bailey and Marcos López de Prado. Balanced Baskets: A New Approach to Trading and Hedging Risks. *Journal of Investment Strategies*, 1(4):21–62, 2012.

Fischer Black, Michael C. Jensen and Myron Scholes. The Capital Asset Pricing Model: Some Empirical Tests. In Michael C. Jensen, editor, *Studies in the Theory of Capital Markets*, pages 79–121. Praeger Publishers Inc., 1972.

Fisher Black and Robert Litterman. Global Portfolio Optimization. *Financial Analysts Journal*, September:28–43, 1992.

John C. Bogle. *The Little Book of Common Sense Investing: The Only Way to Guarantee Your Fair Share of Stock Market Returns (Little Books. Big Profits)*. Wiley, 2007.

Markus K. Brunnermeier and Lasse Heje Pedersen. Market Liquidity and Funding Liquidity. *Review of Financial Studies*, 22(6):2201–2238, 2009.

Roger Clarke, Harindra De Silva and Stephen Thorley. Minimum Variance Portfolio Composition. *Journal of Portfolio Management*, 37(2):31–45, 2011.

Roger Clarke, Harindra De Silva and Stephen Thorley. Minimum Variance, Maximum Diversification and Risk Parity: An Analytic Perspective. *Journal of Portfolio Management*, 39(3):39–53, 2013.

David Cowan and Sam Wilderman. Rethinking Risk: What the Beta Puzzle Tells us About Investing, 2011. GMO White Paper.

Hans Föllmer and Alexander Schied. Convex Risk Measures and Trading Constraints. *Finance and Stochastics*, 6:429–427, 2002.

Hans Föllmer and Alexander Schied. *Stochastic Finance: An Introduction in Discrete Time*. Walter de Gruyter, third edition, 2011.

Andrea Frazzini and Lasse Heje Pedersen. Betting Against Beta, 2010. Working Paper, AQR Capital Management, New York University and NBER Working Paper 16601.

Craig French. The Treynor Capital Asset Pricing Model. *Journal of Investment Management*, 1(2):60–72, 2003.

Fangjian Fu. Idiosyncratic Risk and the Cross-Section of Expected Stock Returns. *Journal of Financial Economics*, 91:24–37, 2009.

John Geanakoplos. The Leverage Cycle, 2010. Cowles Foundation Discussion Paper No.. 1715R.

Lisa R. Goldberg and Ola Mahmoud. Risk Without Return. *Journal of Investment Strategies*, 2(2):111–120, 2013.

Ralph Goldsticker. Volatility Targeting May Miss the Mark, 2012. BNY Mellon Asset Management.

Benjamin Graham. *The Intelligent Investor*. Collins, 1949.

Philippe Jorion. Risk Management Lessons From Long-Term Capital Management. *European Financial Management*, 2000.

Christoph Kessler and Günter Schwarz. Investment management for private investors at SBC - The benchmark, 1996. Swiss Bank Corporation / The Global 2nd Quarter.

Rob Kozlowski. Kansas City Employees pension fund commits to BlackRock for Risk Parity. *Pensions & Investments,* June 20, 2013.

Xi Li, Rodney Sullivan and Luis Garcia-Feijoo. The Limits to Arbitrage: The Low Risk Anomaly. forthcoming in *Financial Analysts Journal*, 2013.

John Lintner. Securities, Prices, Risk and Maximal Gains from Diverification. *The Journal of Finance*, 20(4):13–37, 1965a.

John Lintner. The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets. *Review of Economics and Statistics*, 47(1):587–615, 1965b.

Rudolf Lörtscher. Investment Policy at SBC - basic premises and model, 1990. Swiss Bank Corporation / The Global 10/90.

Ulrike Malmendier, Geoffrey Tate and Jon Yan. Overconfidence and Early-Life Experiences: The Effect of Managerial Traits on Corporate Financial Policies. *Journal of Finance*, 66(5):1687–1783, 2011.

Harry Markowitz. Portfolio Selection. *Journal of Finance*, VII(1):77–91, 1952.

Harry M. Markowitz. Market Efficiency: A Theoretical Distinction and So What? *Financial Analysts Journal*, September/October:17–30, 2005.

Jan Mossin. Equilibrium in a Capital Asset Market. *Econometrica*, 34(4):768–783, 1966.

Lorenzo Naranjo. Implied Interest Rates in a Market with Frictions, 2009. Working paper, New York University.

Edward Qian. Risk Parity Portfolios: Efficient Portfolios Through True Diversification, 2005. PanAgora.

Stephen Ross. *Neoclassical Finance*. Princeton University Press, 2004.

James Sefton, David Jessop, Giuliano De Rossi, Claire Jones and Heran Zhang. Low-Risk Investing, 2011. UBS Investment Research.

Ronnie R. Shah. Understanding Low Volatility Strategies: Minimum Variance, 2011. Dimensional Fund Advisors.

William Sharpe. Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk. *Journal of Finance*, 19(3):425–442, 1964.

Hersh Shefrin and Meir Statman. Behavioral Finance in the Financial Crisis: Market Efficiency, Minsky, and Keynes, 2011. Santa Clara University.

Jack Treynor. Market, Value, Time and Risk, 1961. Unpublished Manuscript 95-209, "Rough Draft" dated 8/8/61.

Jack Treynor. Toward a Theory of Market Value of Risky Assets, 1962. Unpublished Manuscript, "Rough Draft" dated by Mr. Treynor to the fall of 1962. A final version was published in 1999, in *Asset Pricing and Portfolio Performance*, Robert J. Korajczyk (editor) London: RiskBooks, 15–22.

Julia Werdiger. Carlyle Capital's Lenders Liquidating Securities. *New York Times,* March 7, 2008.