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Comparative Statics in Finite Horizon Finance Economies with Stochastic Taxation

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Abstract

This paper studies comparative statics of Financial Markets (FM) equilibria in the finite horizon General Equilibrium with Incomplete Markets (GEI) model with respect to changes in stochastic tax rates imposed on agents' endowments and dividends. We show that under reasonable assumptions, without assuming CRRA and identical agents, an increase in the current dividend tax rate unambiguously reduces current asset prices. The paper also finds that there exists a bound $\overline{B}$ such that for a coefficient of relative risk aversion less than $\overline{B}$, an increase in a future dividend tax rate reduces current price of tradable assets. At the same time, for a coefficient of relative risk aversion greater than $\overline{B}$, an increase in a future dividend tax rate boosts the current price of tradable assets. Finally, for a coefficient of relative risk aversion equal to $\overline{B}$, an increase in a future dividend tax rate leaves current consumption and current price of tradable assets unchanged. As a special case, under additional assumptions, $\overline{B}$ is equal to 1. Also, under reasonable assumptions, an increase in the current endowment tax rate reduces current asset prices, while an increase in a future endowment tax rate boosts current asset prices.

Keywords: Stochastic Taxation, GEI, Equity Premium, Complete Markets, Comparative Statics, Risk Aversion, CCAPM, Property Rights

JEL Classification: D5; D9; E13; G12; H20.

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1. INTRODUCTION

Taxes are a part of individuals' and corporations' budget constraints. Therefore, taxes clearly affect equilibrium commodity and asset prices and allocations. Also, changes in various tax rates, especially income tax, are driven by the constantly changing political balance of power, and the direction of those changes seems to have been anything but predictable. Thus, it seems entirely appropriate to regard future taxation as stochastic.

But if taxation is stochastic, then it is clearly a risk factor affecting equilibrium asset prices through stochastic discount factors and after-tax dividends. Since this risk cannot be eliminated or substantially reduced by diversification, standard finance theory suggests that it ought to be an asset-pricing risk factor, which ought to affect asset prices and allocations.

Surprisingly, however, there has been very little research done to date on the effects of stochastic taxes on equilibrium asset prices and allocations. The research done so far, relies on the CCAPM with identical agents and twice-differentiable utility functions and focuses primarily on resolving the so-called "Equity Premium Puzzle." See Magin (2015a), Edelstein and Magin (2013), DeLong and Magin (2009), Sialm (2009) and (2006). While resolving the Equity Premium Puzzle is critically important for confirming the validity of the Lucas-Rubenstein CCAPM with identical agents, the role of insecure property rights (stochastic taxation) in economic theory is much broader. For example, do Financial Markets (FM) equilibria exist in the finite horizon General Equilibrium of Incomplete Markets (GEI) model with stochastic taxation? Do sufficiently small changes in stochastic tax rates preserve the existence and completeness of FM equilibria? Magin (2015b) finds that under reasonable assumptions, FM equilibria exist for all stochastic tax rates imposed on agents' endowments and dividends except for a closed set of measure zero. Moreover, sufficiently small changes in stochastic taxation preserve the existence and completeness of FM equilibria. The next natural question to ask would be: Does an increase in current and future taxes reduce current prices of tradable assets?

This paper studies comparative statics of FM equilibria in the finite horizon GEI model with respect to changes in stochastic tax rates imposed on agents' endowments and dividends. The Sonnenschein-Mantel-Debreu Theorem states that if we exclude prices close to zero, then no further restrictions other than Continuity, Homogeneity and Walras' Law can be imposed on the aggregate excess demand function of an exchange economy. As a result, comparative statics results are fairly rare in general equilibrium (GE).[1] Given the strong methodological connection between GE and

[1] See sections 17.E-17.G in Mas-Colell, Whinston and Green (1995) for discussion.

GEI, it is not surprising that comparative statics results are also rare in GEI. Here, we develop a technique which we believe to be new, and which is potentially applicable in other situations. We show that, although the sign of a derivative of a complex object of interest may be ambiguous, as is typically the case in GEI, the sign of the derivative of this complex object of interest is always the same as that of the derivative of some other simple and more intuitive object. While the signs of the derivative of this simple and more intuitive object in GEI are often indeterminate, many of them have a natural sign; the presence of the opposite sign is viewed as possible but somewhat rare and pathological. Our methods give intuitive signs to derivatives in cases where the natural sign may not be obvious, by showing that they are the same as the signs in cases where there is an obvious natural sign.

The first major finding of this paper is Theorem 2.2.2. It analyzes comparative statics of current asset prices with respect to changes in current dividend taxes. Dividends are paid in units of a numeraire good. It states that although the sign of a derivative of current equilibrium asset prices with respect to current dividend tax rates may be ambiguous, it is always the same as that of the derivative of the current equilibrium consumption of the numeraire good with respect to current dividend tax rates. Specifically, if the numeraire good is a normal good, then under reasonable assumptions, without assuming CRRA and identical agents, an increase in the current dividend tax rate reduces current asset prices. Otherwise, if the numeraire good is an inferior good, then an increase in the current dividend tax rate boosts current asset prices. While inferior goods have been demonstrated to exist, it is generally believed that they are rare: at any given time and price level, the demand for the vast majority of goods moves in the intuitive direction with respect to changes in after-tax income.[2] So it is reasonable to assume that the numeraire good is a normal good. Therefore, it is natural to conclude that under reasonable assumptions, an increase in the current dividend tax rate reduces current asset prices.

Corollary 2.2.4. of the above theorem states that although the sign of a derivative of before-tax and after-tax rates of return for tradable assets with respect to current dividend tax rates may be ambiguous, it is always the opposite of that of the derivative of the current equilibrium consumption of the numeraire good with respect to current dividend tax rates. Specifically, if the numeraire good is a normal good, then under reasonable assumptions, without assuming CRRA and identical agents, an increase in the current dividend tax rate boosts before-tax and after-tax rates of return for tradable assets. Otherwise, if the numeraire good is an inferior good, then

[2] In his 1957 book "A Theory of the Consumption Function" Milton Friedman develops the Permanent Income Hypothesis and establishes a strong positive correlation between the permanent consumption and the permanent income.

an increase in the current dividend tax rate reduces before-tax and after-tax rates of return for tradable assets. Since it is reasonable to assume that the numeraire good is a normal good, it is natural to conclude that under reasonable assumptions, an increase in the current dividend tax rate boosts before-tax and after-tax rates of return for tradable assets. Assuming that the real risk-free rate of return is constant, we can also conclude that an increase in the current dividend tax rate boosts before-tax and after-tax risk premiums for tradable assets.

The second major finding of this paper is Theorem 2.2.5. It analyzes comparative statics of current asset prices with respect to changes in future dividend taxes. It states that there exists a bound $\overline{B}$ such that for a coefficient of relative risk aversion less than $\overline{B}$, an increase in a future dividend tax rate reduces current prices of tradable assets. At the same time, surprisingly, for a coefficient of relative risk aversion greater than $\overline{B}$, an increase in a future dividend tax rate boosts current prices of tradable assets. Finally, for a coefficient of relative risk aversion equal to $\overline{B}$, an increase in a future dividend tax rate leaves current consumption and current prices of tradable assets unchanged. Corollary 2.2.7. states that, as a special case, under additional assumptions, $\overline{B}$ is equal to 1.

Theorem 2.3.2. analyzes comparative statics of current asset prices with respect to changes in current endowment taxes. It states that although the sign of a derivative of current equilibrium asset prices with respect to current endowment tax rates may be ambiguous, it is always the same as that of the derivative of the current equilibrium consumption of the numeraire good with respect to current endowment tax rates. Specifically, if the numeraire good is a normal good, then under reasonable assumptions, without assuming CRRA and identical agents, an increase in the current endowment tax rate reduces current asset prices. Otherwise, if the numeraire good is an inferior good, then an increase in the current endowment tax rate boosts current asset prices. Since it is reasonable to assume that the numeraire good is a normal good, it is natural to conclude that under reasonable assumptions, an increase in the current endowment tax rate reduces current asset prices.

Theorem 2.3.4. analyzes comparative statics of current asset prices with respect to changes in future endowment taxes. It states that although the sign of a derivative of current equilibrium asset prices with respect to future endowment tax rates may be ambiguous, it is always the opposite to that of the derivative of the future equilibrium consumption of the numeraire good with respect to future endowment tax rates. Specifically, if the numeraire good is a normal good, then under reasonable assumptions, without assuming identical agents, an increase in the future endowment tax rate boosts current asset prices. Otherwise, if the numeraire good is an inferior good, then an increase in the future endowment tax rate reduces current asset prices. Since it is reasonable to assume that the numeraire good is a normal good, it is natural to conclude that under reasonable assumptions, an increase in the future endowment tax rate boosts current asset prices. Theorem 2.3.5. obtains a similar result by assuming identical agents but without necessarily assuming CRRA.

The paper is organized as follows. Section 2 studies comparative statics of FM equilibria with respect to changes in stochastic taxation of endowments and dividends. Section 3 concludes.

2. COMPARATIVE STATICS OF FM EQUILIBRIA WITH STOCHASTIC TAXATION OF DIVIDENDS AND ENDOWMENTS

2.1. Definitions

First, we need to introduce several basic notions to define finite horizon FM Economies with stochastic taxation of endowments and dividends.

DEFINITION: *Let (Ω, Σ, μ) be a probability measure space and $\mathbb{T} = \{0, ..., \overline{T}\}$. Consider a family $\mathbb{F}$ of partitions of Ω given by*

$$\mathbb{F} = \{\mathcal{F}_{t+T} \mid \mathcal{F}_{t+T} \subset \mathcal{P}(\Omega), T \in \mathbb{T}\},$$

where

$$\begin{cases} \mathcal{F}_t = \Omega, \\ \mathcal{F}_{t+\overline{T}} = \{\{\omega\} \mid \omega \in \Omega\}. \end{cases}$$

Then we say that the partition $\mathcal{F}_{t+T+1}$ is finer than the partition $\mathcal{F}_{t+T}$ if

$$\begin{cases} \sigma' \in \mathcal{F}_{t+T} \\ \sigma \in \mathcal{F}_{t+T+1} \end{cases} \Longrightarrow \begin{array}{c} \sigma \subset \sigma' \\ or \\ \sigma \cap \sigma' = \varnothing \end{array}.$$

DEFINITION: *Consider a family $\mathbb{F}$ of partitions of Ω given by*

$$\mathbb{F} = \{\mathcal{F}_{t+T} \mid \mathcal{F}_{t+T} \subset \mathcal{P}(\Omega), T \in \mathbb{T}\},$$

where

$$\begin{cases} \mathcal{F}_t = \Omega, \\ \mathcal{F}_{t+\overline{T}} = \{\{\omega\} \mid \omega \in \Omega\}. \end{cases}$$

Assume that the partition $\mathcal{F}_{t+T+1}$ is finer than the partition $\mathcal{F}_{t+T}$ $\forall T \in \mathbb{T}$. Then we define the event-tree ET as

$$ET = \{\xi = (t+T,\ \sigma) \mid T \in \mathbb{T},\ \sigma \in \mathcal{F}_{t+T}\}.$$

DEFINITION: *Let ET be an event-tree. Then we define function*

$$T : ET \longrightarrow \mathbb{N}_+$$

as

$$T(\xi) = T,$$

where

$$\xi = (t+T,\ \sigma) \in ET.$$

DEFINITION: *Let ET be an event-tree. Then we define the initial node ξ_0 of ET as*

$$\xi_0 = (t,\ \sigma),$$

where $\sigma = \Omega$.

DEFINITION: *Let ET be an event-tree. If $\xi_0 \in ET$ is the initial node of ET, then we define the set ET^+ of non-initial nodes of ET as*

$$ET^+ = ET \setminus \{\xi_0\}.$$

DEFINITION: *Let ET be an event-tree. Then we define a terminal node $\xi_{t+\overline{T}}$ of ET as*

$$\xi_{t+\overline{T}} = \big(t+\overline{T},\ \sigma\big),$$

where $\sigma \in \mathcal{F}_{t+\overline{T}}$.

DEFINITION: *Let ET be an event-tree. Then we define the set $ET_{t+\overline{T}}$ of all terminal nodes of ET as*

$$ET_{t+\overline{T}} = \big\{\big(t+\overline{T},\ \sigma\big) \mid \sigma \in \mathcal{F}_{t+\overline{T}}\big\}.^3$$

DEFINITION: *Let ET be an event-tree. Then we define the set ET^- of all non-terminal nodes of ET as*

[3] If $|ET| = \infty$, then $ET_{t+\overline{T}} = \varnothing$.

$$ET^- = ET \setminus ET_{t+\overline{T}}.$$[4]

DEFINITION: *Let ET be an event-tree. Then $\forall \xi \in ET^-$such that $\xi = (t+T,\ \sigma)$* we *define the set of all immediate successors of ξ as*

$$\begin{aligned}\xi^+ &= \{\xi' \in ET \mid \xi' = (t + T(\xi) + 1,\ \sigma'),\, \sigma' \subset \sigma\}\\ &= \{\xi' \in ET \mid \xi' = (t + T + 1,\ \sigma'),\, \sigma' \subset \sigma\}.\end{aligned}$$

DEFINITION: *Let ET be an event-tree. Then $\forall \xi \in ET^-$ define the number of all immediate successors of ξ called the branching number at ξ as*

$$b(\xi) = \left|\xi^+\right|.$$

DEFINITION: *Let ET be an event-tree. Then we define a binary relation $\geqslant$ on ET as follows. Suppose that*

$$\left\{\begin{array}{c}\xi = (t+T,\ \sigma),\\ \xi' = (t+T',\ \sigma').\end{array}\right.$$

Then

$$\xi' \geqslant \xi \Longleftrightarrow \left\{\begin{array}{c} T(\xi') \geqslant T(\xi),\\ \sigma' \subset \sigma.\end{array}\right. \Longleftrightarrow \left\{\begin{array}{c} T' \geqslant T,\\ \sigma' \subset \sigma.\end{array}\right.$$

We say that ξ' succeeds ξ.

DEFINITION: *Let ET be an event tree. Then $\forall \xi \in ET$ define the subtree $ET(\xi)$ starting at ξ or the set of all successors of ξ as*

$$ET(\xi) = \{\xi' \in ET \mid \xi' \geqslant \xi\}.$$

DEFINITION: *Let ET be an event tree. Then we define a binary relation $>$ on ET as follows. Suppose that*

$$\left\{\begin{array}{c}\xi = (t+T,\ \sigma),\\ \xi' = (t+T',\ \sigma').\end{array}\right.$$

Then

$$\xi' > \xi \Longleftrightarrow \left\{\begin{array}{c} T(\xi') > T(\xi),\\ \sigma' \subset \sigma.\end{array}\right. \Longleftrightarrow \left\{\begin{array}{c} T' > T,\\ \sigma' \subset \sigma.\end{array}\right.$$

[4] If $|ET| = \infty$, then $ET^- = ET$.

We say that ξ' strictly succeeds ξ.

Clearly,

$$\xi' > \xi \Longleftrightarrow \begin{cases} \xi' \geqslant \xi, \\ \xi' \neq \xi. \end{cases}$$

DEFINITION: *Let ET be an event-tree. Then $\forall \xi \in ET$ define the set of all strict successors of ξ as*

$$ET^{+}(\xi) = \{\xi' \in ET(\xi) \mid \xi' > \xi\}.$$

DEFINITION: *Let ET be an event-tree. Then $\forall \xi \in ET$ define the set of all non-terminal successors of ξ as*

$$ET^{-}(\xi) = \{\xi' \in ET(\xi) \mid \xi' \in ET^{-}\}.$$

Suppose there is an event-tree ET and a set I of finitely living investors-consumers who trade a set L of commodities on spot markets and a set K of assets on financial markets.

DEFINITION: *We define the before-tax individual endowment e_i of agent $i \in I$ as*

$$e_i = \{e_i(\xi,\ l)\}_{(\xi,\ l) \in ET \times L} \in \mathbb{R}_{+}^{|ET \times L|}.$$

Set also

$$e_i(\xi) = \{e_i(\xi,\ l)\}_{l \in L} \in \mathbb{R}_{+}^{|L|}\ \forall (\xi,\ i) \in ET \times I.$$

DEFINITION: *We define the matrix of before-tax individual endowments e as*

$$e = \{e_i\}_{i \in I} \in \mathbb{R}_{+}^{|ET \times L \times I|}.$$

DEFINITION: *We define the consumption of agent $i \in I$ as*

$$c_i = \{c_i(\xi,\ l)\}_{(\xi,\ l) \in ET \times L} \in \mathbb{R}_{+}^{|ET \times L|}.$$

Set also

$$c_i(\xi) = \{c_i(\xi,\ l)\}_{l \in L} \in \mathbb{R}_{+}^{|L|}\ \forall (\xi,\ i) \in ET \times I.$$

DEFINITION: *We define the vector of spot prices as*

$$p = \{p(\xi,\ l)\}_{(\xi,\ l) \in ET \times L} \in \mathbb{R}^{|ET \times L|}$$

such that

$$p(\xi,\, 1) = 1\ \forall \xi \in ET.$$

Set also

$$p(\xi) = \{p(\xi,\ l)\}_{l \in L} \in \mathbb{R}^{|L|}\ \forall \xi \in ET.$$

We are now ready to discuss the asset structure of our model.

DEFINITION: *Let $\xi(k) \in ET$ be the node of issue for an asset $k \in K$. Define the set ζ of all nodes of issue of existing financial contracts*

$$\zeta = \{\xi(k) \mid k \in K\}.$$

DEFINITION: *We define the matrix of real before-tax dividends d paid in units of the numeraire good 1 as*

$$d = \{d(\xi,\ k)\}_{(\xi,\ k) \in ET \times K} \in \mathbb{R}^{|ET \times K|},$$

where

$$d(\xi,\, k) = 0\ \ \forall \xi \in ET \setminus ET^{+}(\xi(k))\ \forall k \in K,$$

i.e., an asset $k \in K$ issued at node $\xi(k) \in ET$ pays no dividends prior to or at node $\xi(k) \in ET$.[5]

Set also

$$d(\xi) = \{d(\xi,\ k)\}_{k \in K} \in \mathbb{R}^{|K|}\ \forall \xi \in ET.$$

DEFINITION: *We define the space of asset dividends as*

$$\mathcal{D} = \mathbb{R}^{|ET \times K|}.$$

DEFINITION: *We define the set of all actively traded financial contracts $K(\xi) \subset K$ at node $\xi \in ET$ as*

[5] The model could be generalized for the case, where dividends are paid in bundles of all $|L|$ goods, not just in units of the numeraire good 1. See Duffie and Shafer (1986), for example. In that case $d = \{d(\xi,\ k,\ l)\}_{(\xi,\ k,\ l) \in ET \times K \times L} \in \mathbb{R}^{|ET \times K \times L|}$.

$$K(\xi) =$$
$$= \{k \in K \mid \underbrace{\xi \in ET\left(\xi(k)\right)}_{(1)}, \underbrace{\exists \xi' \in ET^{+}(\xi) \ s.t. \ d(\tau_d(\xi', \ k)) \neq 0}_{(2)}\} \subset K.$$

That is, the set of financial contracts which are actively traded at node $\xi \in ET$ consists of those financial contracts which (1) have been issued prior to or at node $\xi \in ET$ and (2) whose expiration node strictly follows node $\xi \in ET$, since there is no point in trading a security which never yields a dividend.

DEFINITION: *We define the matrix of asset prices as*

$$q = \left\{q(\xi, \ k)\right\}_{(\xi, \ k) \in ET \times K} \in \mathbb{R}^{|ET \times K|},$$

where

$$q(\xi, \, k) = 0 \ \forall \xi \in ET \ s.t. \ k \in K \setminus K(\xi), \, \forall k \in K,$$

i.e., $q(\xi, \, k) = 0$ if an asset $k \in K$ is not actively traded at node $\xi \in ET$.[6]
Set also

$$q(\xi) = \left\{q(\xi, \ k)\right\}_{k \in K} \in \mathbb{R}^{|K|}.$$

DEFINITION: *We define the space of asset prices as*

$$\mathcal{Q} = \mathbb{R}^{|ET \times K|}.$$

DEFINITION: *We define an asset portfolio z_i held by agent $i \in I$ as follows*

$$z_i = \left\{z_i(\xi, \ k)\right\}_{(\xi, \ k) \in ET \times K} \in \mathbb{R}^{|ET \times K|},$$

where

$$z_i(\xi, \, k) = 0 \ \forall \xi \in ET \ s.t. \ k \in K \setminus K(\xi), \, \forall k \in K,$$

i.e., $z_i(\xi, \, k) = 0$ if an asset $k \in K$ is not actively traded at node $\xi \in ET$.[7]
Set also

[6] If $|ET| < \infty$, then $ET_{t+\overline{T}} \neq \varnothing$ and $K(\xi) = \varnothing \ \forall \xi \in ET_{t+\overline{T}}$, since there is no point in trading a security which will never yield a dividend.

[7] If $|ET| < \infty$, then $ET_{t+\overline{T}} \neq \varnothing$ and $K(\xi) = \varnothing \ \forall \xi \in ET_{t+\overline{T}}$, since there is no point in trading a security which will never yield a dividend.

$$z_i(\xi) = \{z_i(\xi,\ k)\}_{k \in K} \in \mathbb{R}^{|K|}.$$

DEFINITION: *We define the portfolio space as*

$$\mathcal{Z} = \mathbb{R}^{|ET \times K|}.$$

We need to introduce now several definitions to incorporate stochastic taxation $\tau = (\tau_e, \tau_d)$ imposed on agents' endowments and assets' dividends and used to finance government spending G into the General Equilibrium Theory of Financial Markets.

DEFINITION: *We define the individual endowment tax* τ_{e_i} *imposed on the individual endowment* e_i *of agent* $i \in I$ *as*

$$\tau_{e_i} = \{\tau_{e_i}(\xi,\ l)\}_{(\xi,\ l) \in ET \times L} \in [0, 1[^{|ET \times L|}.$$

Set also

$$\tau_{e_i}(\xi) = \{\tau_{e_i}(\xi,\ l)\}_{l \in L} \in \mathbb{R}_+^{|L|}\ \forall\, (\xi,\ i) \in ET \times I.$$

DEFINITION: *We define the matrix of individual endowment taxes* τ_e *as*

$$\tau_e = \{\tau_{e_i}\}_{i \in I} \in [0, 1[^{|ET \times L \times I|}.$$

It is reasonable to assume that the sizes of individual endowments e_i are decreasing functions $e_i(\tau_{e_i})$ of individual endowment tax rates τ_{e_i}.[8] Moreover, since future endowment tax rates are uncertain, it is reasonable to view the taxation of individual endowments as stochastic. So set

$$e_i = e_i(\tau_{e_i})\ \forall i \in I,$$
$$e = \{e_i(\tau_{e_i})\}_{i \in I}.$$

DEFINITION: *We define the after-tax individual endowment* $(1 - \tau_{e_i}) \cdot e_i(\tau_{e_i})$ *of agent* $i \in I$ *as*

$$(1 - \tau_{e_i})\, e_i(\tau_{e_i}) = \{(1 - \tau_{e_i}(\xi,\ l)) \cdot e_i(\tau_{e_i}(\xi,\ l))\}_{(\xi,\ l) \in ET \times L} \in \mathbb{R}_+^{|ET \times L|}.$$

Set also

$$(1 - \tau_{e_i}(\xi))\, e_i(\tau_{e_i}(\xi)) = \{(1 - \tau_{e_i}(\xi,\ l)) \cdot e_i(\tau_{e_i}(\xi,\ l))\}_{\ l \in L} \in \mathbb{R}_+^{|L|}\ \forall \xi \in ET.$$

[8] See Kawano (2013), for example, for a review of the Dividend Clientele Hypothesis.

DEFINITION: *We define the matrix of after-tax individual endowments* $(1-\tau)\,e(\tau_e)$ *as*

$$(1-\tau_e)\,e(\tau_e) = \{(1-\tau_{e_i})\,e_i(\tau_{e_i})\}_{i\in I} \in \mathbb{R}_+^{|ET\times L\times I|}.$$

DEFINITION: *We define the dividend tax* τ_d *imposed on assets' dividends* d *as*

$$\tau_d = \{\tau_d(\xi,\ k)\}_{(\xi,\ k)\in ET\times K} \in [0,\ 1]^{|ET\times K|}.$$

Set also

$$\tau_d(\xi) = \{\tau_d(\xi,\ k)\}_{k\in K} \in [0,1]^{|K|}\ \forall\,(\xi,\ k) \in ET\times K.$$

Consistent with the Dividend Clientele Hypothesis (DCH), it is reasonable to assume that assets' dividends are decreasing functions $d(\tau_d)$ of dividend tax rates τ_d.[9] Moreover, since future dividend tax rates are uncertain, it is reasonable to view the taxation of dividends as stochastic. So set

$$d(\xi,\,k) = d(\xi,\,k,\,\tau_d)\ \forall\,(\xi,\ k) \in ET\times K.$$

DEFINITION: *We define real after-tax dividends paid in units of the numeraire good* 1 *as*

$$(1-\tau_d)\,d(\tau_d) = \{(1-\tau_d(\xi,\ k))\cdot d(\xi,\ k,\ \tau_d)\}_{(\xi,\ k)\in ET\times K} \in \mathbb{R}^{|ET\times K|}.$$

Set also

$$(1-\tau_d\,(\xi))\,d(\xi,\,\tau_d) = \{(1-\tau_d(\xi,\ k))\cdot d(\xi,\ k,\ \tau_d)\}_{\,k\in K} \in \mathbb{R}^{|K|}\ \forall\xi \in ET.$$

DEFINITION: *Let* ζ *be the set of all nodes of issue of* $|K|$ *existing financial contracts and* d *be the* $|ET\times K|$ *matrix of dividends. Then we call the pair*

$$\mathcal{A}(\tau_d) = (\zeta,\,(1-\tau_d)\cdot d(\tau_d))$$

the financial structure.

We are now ready to define a finite horizon FM Economy with stochastic taxation.

[9] See Kawano (2013), for example, for a review of the Dividend Clientele Hypothesis.

DEFINITION: *We denote by* $\mathcal{E}(ET,\,(1-\tau_e)\cdot e\,(\tau_e)\,,\,\succeq,\,\mathcal{A}(\tau_d))$ *with*

$$\mathcal{A}(\tau_d) = (\zeta,\,(1-\tau_d)\cdot d(\tau_d))$$

a finite horizon FM Economy with stochastic taxation

$$\tau = (\tau_e,\,\tau_d) \in [0,1[^{|ET\times L\times I|} \times [0,\;1]^{|ET\times K|},$$

where

$$|ET| < \infty,\,|I| < \infty,\,|L| < \infty,\,|K| < \infty,$$

agents' preferences $\succeq_i$ *are given by the utility function*

$$U_i : \mathbb{R}_+^{|E\mathbb{T}\times\mathbb{L}|} \times \mathbb{R}_+^{|E\mathbb{T}\times\mathbb{L}|} \longrightarrow \mathbb{R}$$

such that

$$U_i(c_i,\,G) = \sum_{(\xi,\,l)\in ET\times L} \Pr(\xi)\cdot b_i^{T(\xi)}\cdot[u_i(c_i(\xi,\;l)) + v_i(G(\xi,\;l))]\ \forall i\in I,$$

where the government spending

$$G = \{G(\xi,\;l)\}_{(\xi,\,l)\in ET\times L} \in \mathbb{R}_+^{|E\mathbb{T}\times\mathbb{L}|}$$

is given by

$$G(\xi,\,l) = \begin{cases} \sum_{i\in I}\tau_{e_i}(\xi,\,l)\cdot e_i(\xi,\,l)\ \forall(\xi,\,l)\in ET\times[L\setminus\{1\}] \\ \left[\sum_{i\in I}\tau_{e_i}(\xi,\;l)\cdot e_i(\xi,\;l) + \sum_{k\in K}\overline{z}(k)\cdot\tau_d(\xi,\;k)\cdot d(\xi,\;k)\right]\ \forall(\xi,\,l)\in ET\times\{1\} \end{cases}$$

and $\overline{z}(k)$ *is the total number of outstanding shares of asset* $k\in K$.

Following Magill and Quinzii (1996), define matrix $W(q(\tau),\,(1-\tau_d)\cdot d(\tau_d))$

$$\forall(q(\tau),\,(1-\tau_d)\cdot d(\tau_d)) \in Q\times\mathbb{R}^{|ET\times K|},$$

which will significantly simplify writing of agents' budget constraints.

DEFINITION: *Let $\mathcal{E}(ET,\ (1-\tau_e)\cdot e(\tau_e),\ \succeq,\ \mathcal{A}(\tau_d))$ be a finite horizon FM Economy with stochastic taxation*

$$\tau = (\tau_e,\, \tau_d) \in [0,1[^{|ET\times L\times I|} \times [0,\ 1]^{|ET\times K|}.$$

Define the $ET \times ET$ payoff matrix matrix $W(q(\tau),\, (1-\tau_d)\cdot d(\tau_d))$

$$\forall (q(\tau),\, (1-\tau_d)\cdot d(\tau_d)) \in Q \times \mathbb{R}^{|ET\times K|}$$

as

$$W_{\xi,\,\xi^+}(q(\tau),\, (1-\tau_d)\cdot d(\tau_d)) = q(\xi^+) + \left(1-\tau_d(\xi^+)\right)\cdot d(\tau_d(\xi^+)),$$
$$W_{\xi,\,\xi}(q(\tau),\, (1-\tau_d)\cdot d(\tau_d)) = -q(\xi),$$
$$W_{\xi,\,\xi'}(q(\tau),\, (1-\tau_d)\cdot d(\tau_d)) = 0\ \forall \xi' \notin \xi^+,\, \xi' \neq \xi.$$

Matrix $W(q(\tau), (1-\tau_d) \cdot d(\tau_d))$

$\lvert K \rvert$ Columns for ξ_0		$\lvert K \rvert$ Columns for ξ^-	$\lvert K \rvert$ Columns for ξ		
$-q(\xi_0, \tau)$	0	0	0	0	ξ_0
$q(\xi_0^+, \tau) + \left(1-\tau_d(\xi_0^+)\right) \cdot d(\tau_d(\xi_0^+))$	$\dots$	$\dots$	0	0	ξ_0^+
0	$\dots$	$\dots$	0	0	
0	0	$q(\xi, \tau) + (1-\tau_d(\xi)) \cdot d(\tau_d(\xi))$	$-q(\xi, \tau)$	0	ξ
0	$\dots$	$\dots$	0	$\dots$	
0	0	0	$q(\xi^+, \tau) + (1-\tau_d(\xi^+)) \cdot d(\tau_d(\xi^+))$	$\dots$	ξ^+
0	0	0	0	$\dots$	

DEFINITION: *Let $\mathcal{E}(ET,\ (1-\tau_e)\cdot e\left(\tau_e\right),\ \succeq,\ \mathcal{A}(\tau_d))$ be a finite horizon FM Economy with stochastic taxation*

$$\tau=(\tau_e,\ \tau_d)\in[0,\ 1]^{|ET\times L\times I|}\times[0,\ 1]^{|ET\times K|}.$$

Then define the budget set of agent $i\in I$ as follows

$$\begin{gathered}B(p,\ q,\ (1-\tau_{e_i})\cdot e_i(\tau_{e_i}),\ \mathcal{A}(\tau_d))=\\ \left\{c_i\in E_+\ \middle|\ \begin{array}{c}\exists z_i\in\mathcal{Z}\ \text{such that}\ p\left(\xi\right)\cdot c_i\left(\xi\right)+q\left(\xi\right)\cdot z_i\left(\xi\right)\leqslant\\ \leqslant p\left(\xi\right)\cdot\left(1-\tau_{e_i}\left(\xi\right)\right)\cdot e_i\left(\xi\right)+\left(q\left(\xi\right)+\left(1-\tau_d\left(\xi\right)\right)\cdot d\left(\xi\right)\right)\cdot z_i\left(\xi^-\right)\ \forall\xi\in ET\end{array}\right\}=\\ =\\ \{c_i\in E_+\,|\exists z_i\in\mathcal{Z}\ \text{such that}\ p\cdot c_i-p\cdot(1-\tau_{e_i})\cdot e_i(\tau_{e_i})=W(q(\tau_e,\ \tau_d),\ (1-\tau_d)\cdot d(\tau_d))\cdot z_i\}.\end{gathered}$$

DEFINITION: *An FM equilibrium for an FM economy $\mathcal{E}(ET,\ (1-\tau_e)\cdot e(\tau_e),\ \succeq,\ \mathcal{A}\left(\tau_d\right))$ with stochastic taxation*

$$\tau=(\tau_e,\ \tau_d)\in[0,1[^{|ET\times L\times I|}\times[0,\ 1]^{|ET\times K|}$$

is a pair

$$(\{(\overline{c}_i(\tau),\ \overline{z}_i(\tau))\}_{i\in I},\ \ (\overline{p}(\tau),\ \overline{q}(\tau)))\in\mathbb{R}_+^{|ET\times L\times I|}\times\mathcal{Z}^{|I|}\times\mathbb{R}_+^{|ET\times L|}\times Q$$

such that

$$\begin{gathered}(\overline{c}_i(\tau),\ \overline{z}_i(\tau))\in\\ \in\arg\max\left\{U_i(c_i,\ G)\mid(c_i,\ z_i)\in B(\overline{p}(\tau),\ \overline{q}(\tau),\ \ (1-\tau_{e_i})\cdot e_i(\tau_{e_i}),\ \mathcal{A}\left(\tau_d\right))\right\}\ \forall i\in I,\\ \sum_{i\in I}\overline{c}_i(\tau)=\sum_{i\in I}\left(1-\tau_{e_i}\right)\cdot e_i(\tau_e)\end{gathered}$$

and

$$\sum_{i\in I}\overline{z}_i(\tau)=0.$$

DEFINITION: *An FM equilibrium for an FM economy* $\mathcal{E}(ET,\, (1-\tau_e)\cdot e(\tau_e),\, \succeq,\, \mathcal{A}(\tau_d))$ *with stochastic taxation*

$$\tau = (\tau_e,\, \tau_d) \in [0,\ 1]^{|ET\times L\times I|} \times [0,\ 1]^{|ET\times K|}$$

and with assets in positive supply is a pair

$$\left(\left\{(\overline{c}_i(\tau),\ \overline{z}_i(\tau))\right\}_{i\in I},\ \ (\overline{p}(\tau),\ \overline{q}(\tau))\right) \in \mathbb{R}_+^{|ET\times L\times I|} \times \mathcal{Z}^{|I|} \times \mathbb{R}_+^{|ET\times L|} \times Q$$

such that

$$(\overline{c}_i(\tau),\, \overline{z}_i(\tau)) \in$$
$$\in \arg\max\left\{U_i(c_i,\ G) \mid (c_i,\ z_i) \in B(\overline{p}(\tau),\ \overline{q}(\tau),\ \ (1-\tau_{e_i})\cdot e_i(\tau_{e_i}),\ \mathcal{A}(\tau_d))\right\}\ \forall i\in I,$$
$$\sum_{i\in I}\overline{c}_i(\xi,\, 1,\, \tau) = \sum_{i\in I}(1-\tau_{e_i}(\xi,\ 1))\cdot e_i(\xi,\, 1,\, \tau_{e_i}) + \sum_{k\in K}(1-\tau_d(\xi,\ k))\cdot d(\xi,\, k,$$
$$\tau_d)\cdot \overline{z}(k),$$
$$\sum_{i\in I}\overline{c}_i(\xi,\, l,\, \tau) = \sum_{i\in I}(1-\tau_{e_i}(\xi,\ l))\cdot e_i(\xi,\, l,\, \tau_{e_i})\ \forall l\in L\setminus\{1\}$$

and

$$\sum_{i\in I}\overline{z}_i(\tau) = \overline{z} > 0.$$

DEFINITION: *Let* $\mathcal{E}(ET,\, (1-\tau_e)\cdot e\,(\tau_e)\,,\, \succeq,\, \mathcal{A}(\tau_d))$ *be an FM economy with stochastic taxation*

$$\tau = (\tau_e,\, \tau_d) \in [0, 1[^{|ET\times L\times I|} \times [0,\ 1]^{|ET\times K|}.$$

Given

$$(q(\tau),\, (1-\tau_d)\cdot d(\tau_d)) \in Q\ \ \times \mathbb{R}^{|ET\times K|},$$

we say that the No Arbitrage Condition (NAC) holds if

$$rank[W(q(\tau),\, (1-\tau_d)\cdot d(\tau_d))] \leqslant \mid ET\mid -1$$

or, equivalently, there exists a price vector

$$\pi(\tau) = \{\pi(\xi',\ \tau)\}_{\xi'\in ET} \in \mathbb{R}_{++}^{|ET|}$$

such that

$$\pi(\tau)\cdot W(q(\tau),\, (1-\tau_d)\cdot d(\tau_d)) = 0,$$

i.e.,

$$q\,(\xi,\ \tau) = \sum_{\xi'\in ET^+(\xi)} \tfrac{\overline{\pi}(\xi',\, \tau)}{\overline{\pi}(\xi,\, \tau)}\cdot(1-\tau_d\,(\xi'))\cdot d\,(\xi',\ \tau_d)\ \forall\xi\in ET.$$

DEFINITION: *Let $\mathcal{E}(ET,\ (1-\tau_e)\cdot e\left(\tau_e\right),\ \succeq,\ \mathcal{A}(\tau_d))$ be an FM economy with stochastic taxation*

$$\tau=(\tau_e,\ \tau_d)\in[0,1[^{|ET\times L\times I|}\times[0,\ 1]^{|ET\times K|}.$$

Given

$$(q(\tau),\ (1-\tau_d)\cdot d(\tau_d))\in Q\ \times\mathbb{R}^{|ET\times K|},$$

the FM are complete if

$$rank[W(q(\tau),\ (1-\tau_d)\cdot d(\tau_d))]=\mid ET\mid -1$$

or, equivalently, there exists a unique price vector

$$\overline{\pi}(\tau)=\{\overline{\pi}(\xi',\ \tau)\}_{\xi'\in ET}\in\mathbb{R}^{|ET|}_{++}$$

such that

$$\overline{\pi}(\tau)\cdot W(q(\tau),\ (1-\tau_d)\cdot d(\tau_d))=0,$$

i.e.,

$$q\left(\xi,\ \tau\right)=\sum_{\xi'\in ET^{+}(\xi)}\frac{\overline{\pi}(\xi',\,\tau)}{\overline{\pi}(\xi,\,\tau)}\cdot\left(1-\tau_d\left(\xi'\right)\right)\cdot d\left(\xi',\ \tau_d\right)\ \forall\xi\in ET.$$

For existence of FM equilibria in the finite horizon GEI model see, for example, Magill and Quinzii (1996), Magill and Shafer (1991) and (1990), Duffie and Shafer (1986).

We also need to introduce matrix notations for partial derivatives of $\overline{c}$, d, τ_d, $\overline{q}$, $\overline{\pi}$ and τ_{e_i} with respect to the dividend tax $\tau_d\in[0,\ 1]^{|ET\times K|}$.

DEFINITION: *Define the matrix of partial derivatives of the consumption $\overline{c}_i\in\mathbb{R}^{|ET\times L|}$ of an agent $i\in I$ with respect to the dividend tax $\tau_d\in[0,\ 1]^{|ET\times K|}$ as*

$$\frac{\partial\overline{c}_i}{\partial\tau_d}=\left\{\left\{\frac{\partial\overline{c}_i(\xi',\,l,\,\tau)}{\partial\tau_d(\xi,\,k)}\right\}_{(k,\,l)\in K\times L}\right\}_{(\xi',\,\xi)\in ET\times ET}\in\mathbb{R}^{|ET\times ET\times K\times L|}.$$

Set also

$$\frac{\partial\overline{c}_i(\xi',\,\tau)}{\partial\tau_d(\xi)}=\left\{\frac{\partial\overline{c}_i(\xi',\,l,\,\tau)}{\partial\tau_d(\xi,\,k)}\right\}_{(k,\,l)\in K\times L}\in\mathbb{R}^{|K\times L|}.$$

Clearly,

$$\frac{\partial\overline{c}_i}{\partial\tau_d}=\left\{\frac{\partial\overline{c}_i(\xi',\,\tau)}{\partial\tau_d(\xi)}\right\}_{(\xi',\,\xi)\in ET\times ET}.$$

DEFINITION: *Define the matrix of partial derivatives of asset dividends* $d \in \mathbb{R}^{|ET \times K|}$ *with respect to the dividend tax* $\tau_d \in [0,\ 1]^{|ET \times K|}$ as

$$\frac{\partial d}{\partial \tau_d} = \left\{ \left\{ \frac{\partial d(\xi',\, k_1,\, \tau_d)}{\partial \tau_d(\xi,\, k_2)} \right\}_{(k_1,\, k_2,) \in K \times K} \right\}_{(\xi',\, \xi) \in ET \times ET} \in \mathbb{R}^{|ET \times ET \times K \times K|}.$$

Set also

$$\frac{\partial d(\xi',\, \tau_d)}{\partial \tau_d(\xi)} = \left\{ \frac{\partial d(\xi',\, k_1,\, \tau_d)}{\partial \tau_d(\xi,\, k_2)} \right\}_{(k_1,\, k_2,) \in K \times K} \in \mathbb{R}^{|K \times K|}.$$

Clearly,

$$\frac{\partial d}{\partial \tau_d} = \left\{ \frac{\partial d(\xi',\, \tau_d)}{\partial \tau_d(\xi)} \right\}_{(\xi',\, \xi) \in ET \times ET}.$$

DEFINITION: *Define the matrix of partial derivatives of the dividend tax* $\tau_d \in [0,\ 1]^{|ET \times K|}$ *with respect to the dividend tax* $\tau_d \in [0,\ 1]^{|ET \times K|}$ *itself as*

$$\frac{\partial \tau_d}{\partial \tau_d} = \left\{ \left\{ \frac{\partial \tau_d(\xi',\, k_1)}{\partial \tau_d(\xi,\, k_2)} \right\}_{(k_1,\, k_2,) \in K \times K} \right\}_{(\xi',\, \xi) \in ET \times ET} \in \mathbb{R}^{|ET \times ET \times K \times K|}.$$

Set also

$$\frac{\partial \tau_d(\xi')}{\partial \tau_d(\xi)} = \left\{ \frac{\partial d(\xi',\, k_1)}{\partial \tau_d(\xi,\, k_2)} \right\}_{(k_1,\, k_2,) \in K \times K} \in \mathbb{R}^{|K \times K|}.$$

Clearly,

$$\frac{\partial \tau_d}{\partial \tau_d} = \left\{ \frac{\partial \tau_d(\xi')}{\partial \tau_d(\xi)} \right\}_{(\xi',\, \xi) \in ET \times ET}.$$

DEFINITION: *Define the matrix of partial derivatives of asset prices* $\overline{q} \in \mathbb{R}^{|ET \times K|}$ *with respect to the dividend tax* $\tau_d \in [0,\ 1]^{|ET \times K|}$ as

$$\frac{\partial \overline{q}}{\partial \tau_d} = \left\{ \left\{ \frac{\partial \overline{q}(\xi,\, k_1,\, \tau)}{\partial \tau_d(\xi',\, k_2)} \right\}_{(k_1,\, k_2,) \in K \times K} \right\}_{(\xi',\, \xi) \in ET \times ET} \in \mathbb{R}^{|ET \times ET \times K \times K|}.$$

Set also

$$\frac{\partial \overline{q}(\xi,\, \tau)}{\partial \tau_d(\xi')} = \left\{ \frac{\partial \overline{q}(\xi,\, k_1,\, \tau)}{\partial \tau_d(\xi',\, k_2)} \right\}_{(k_1,\, k_2,) \in K \times K} \in \mathbb{R}^{|K \times K|}.$$

Clearly,

$$\frac{\partial \overline{q}}{\partial \tau_d} = \left\{ \frac{\partial \overline{q}(\xi,\, \tau)}{\partial \tau_d(\xi')} \right\}_{(\xi',\, \xi) \in ET \times ET}.$$

DEFINITION: *Define the matrix of partial derivatives of node prices* $\overline{\pi} \in \mathbb{R}^{|ET \times L|}$ *with respect to dividend tax* $\tau_d \in [0,\ 1]^{|ET \times K|}$ *as*

$$\frac{\partial \overline{\pi}}{\partial \tau_d} = \left\{ \left\{ \frac{\partial \overline{\pi}(\xi',\, l,\, \tau)}{\partial \tau_d(\xi,\, k)} \right\}_{(k,\, l) \in K \times L} \right\}_{(\xi',\, \xi) \in ET \times ET} \in \mathbb{R}^{|ET \times ET \times K \times L|}.$$

Set also

$$\frac{\partial \overline{\pi}(\xi',\, \tau)}{\partial \tau_d(\xi)} = \left\{ \frac{\partial \overline{\pi}(\xi',\, l,\, \tau)}{\partial \tau_d(\xi,\, k)} \right\}_{(k,\, l) \in K \times L} \in \mathbb{R}^{|K \times L|}.$$

Clearly,

$$\frac{\partial \overline{\pi}}{\partial \tau_d} = \left\{ \frac{\partial \overline{\pi}(\xi',\, \tau)}{\partial \tau_d(\xi)} \right\}_{(\xi',\, \xi) \in ET \times ET}.$$

DEFINITION: *Define the matrix of partial derivatives of asset dividends* $d \in \mathbb{R}^{|ET \times K|}$ *with respect to the endowment tax* $\tau_{e_i} \in [0,\ 1]^{|ET \times L|}$ *as*

$$\frac{\partial \tau_{e_i}}{\partial \tau_d} = \left\{ \left\{ \frac{\partial \tau_{e_i}(\xi',\, l)}{\partial \tau_d(\xi,\, k)} \right\}_{(k,\, l) \in K \times L} \right\}_{(\xi',\, \xi) \in ET \times ET} \in \mathbb{R}^{|ET \times ET \times K \times L|}.$$

Set also

$$\frac{\partial \tau_{e_i}(\xi')}{\partial \tau_d(\xi)} = \left\{ \left\{ \frac{\partial \tau_{e_i}(\xi',\, l)}{\partial \tau_d(\xi,\, k)} \right\} \right\}_{(k,\, l) \in K \times L} \in \mathbb{R}^{|K \times L|}.$$

Clearly,

$$\frac{\partial \tau_{e_i}}{\partial \tau_d} = \left\{ \frac{\partial \tau_{e_i}(\xi')}{\partial \tau_d(\xi)} \right\}_{(\xi',\, \xi) \in ET \times ET}.$$

Similarly, we need to introduce matrix notations for partial derivatives of $\overline{c}$, d, τ_d, $\overline{q}$, $\overline{\pi}$,and $\tau_{e_i}$with respect to the endowment tax $\tau_{e_i} \in [0,\ 1]^{|ET \times L|}$ of an agent $i \in I$.

DEFINITION: *Define the matrix of partial derivatives of the consumption* $\overline{c}_i \in \mathbb{R}^{|ET \times L|}$ *of an agent* $i \in I$ *with respect to the endowment tax* $\tau_{e_i} \in [0,\ 1]^{|ET \times L|}$ *of an agent* $i \in I$ *as*

$$\frac{\partial \overline{c}_i}{\partial \tau_{e_i}} = \left\{ \left\{ \frac{\partial \overline{c}_i(\xi',\, l_1,\, \tau)}{\partial \tau_{e_i}(\xi,\, l_2)} \right\}_{(l_1,\, l_2,) \in L \times L} \right\}_{(\xi',\, \xi) \in ET \times ET} \in \mathbb{R}^{|ET \times ET \times L \times L|}.$$

Set also

$$\frac{\partial \overline{c}_i(\xi',\, \tau)}{\partial \tau_{e_i}(\xi)} = \left\{ \frac{\partial \overline{c}_i(\xi',\, l_1,\, \tau)}{\partial \tau_{e_i}(\xi,\, l_2)} \right\}_{(l_1,\, l_2,) \in L \times L} \in \mathbb{R}^{|L \times L|}.$$

Clearly,

$$\frac{\partial \overline{c}}{\partial \tau_{e_i}} = \left\{ \frac{\partial \overline{c}_i(\xi',\, \tau)}{\partial \tau_{e_i}(\xi)} \right\}_{(\xi',\, \xi) \in ET \times ET}.$$

DEFINITION: *Define the matrix of partial derivatives of asset dividends* $d \in \mathbb{R}^{|ET \times K|}$ *with respect to the endowment tax* $\tau_{e_i} \in [0,\ 1]^{|ET \times L|}$ *of an agent* $i \in I$ *as*

$$\frac{\partial d}{\partial \tau_{e_i}} = \left\{ \left\{ \frac{\partial d(\xi',\, k,\, \tau_d)}{\partial \tau_{e_i}(\xi,\, l)} \right\}_{(k,\, l) \in K \times L} \right\}_{(\xi',\, \xi) \in ET \times ET} \in \mathbb{R}^{|ET \times ET \times K \times L|}.$$

Set also

$$\frac{\partial d(\xi',\, \tau_d)}{\partial \tau_{e_i}(\xi)} = \left\{ \frac{\partial d(\xi',\, k,\, \tau_d)}{\partial \tau_{e_i}(\xi,\, l)} \right\}_{(k,\, l,) \in K \times L} \in \mathbb{R}^{|K \times L|}.$$

Clearly,

$$\frac{\partial d}{\partial \tau_{e_i}} = \left\{ \frac{\partial d(\xi',\, \tau_d)}{\partial \tau_{e_i}(\xi)} \right\}_{(\xi',\, \xi) \in ET \times ET}.$$

DEFINITION: *Define the matrix of partial derivatives of dividend taxes* $\tau_d \in \mathbb{R}^{|ET \times K|}$ *with respect to the endowment tax* $\tau_{e_i} \in [0,\ 1]^{|ET \times L|}$ of *an agent* $i \in I$ *as*

$$\frac{\partial \tau_d}{\partial \tau_{e_i}} = \left\{ \left\{ \frac{\partial \tau_d(\xi',\, k)}{\partial \tau_{e_i}(\xi,\, l)} \right\}_{(k,\, l) \in K \times L} \right\}_{(\xi',\, \xi) \in ET \times ET} \in \mathbb{R}^{|ET \times ET \times K \times L|}.$$

Set also

$$\frac{\partial \tau_d(\xi')}{\partial \tau_{e_i}(\xi)} = \left\{ \frac{\partial \tau_d(\xi',\, k)}{\partial \tau_{e_i}(\xi,\, l)} \right\}_{(k,\, l,) \in K \times L} \in \mathbb{R}^{|K \times L|}.$$

Clearly,

$$\frac{\partial d}{\partial \tau_{e_i}} = \left\{ \frac{\partial \tau_d(\xi')}{\partial \tau_{e_i}(\xi)} \right\}_{(\xi',\, \xi) \in ET \times ET}.$$

DEFINITION: *Define the matrix of partial derivatives of asset prices* $\overline{q} \in \mathbb{R}^{|ET \times K|}$ *with respect to the endowment tax* $\tau_{e_i} \in [0,\ 1]^{|ET \times L|}$ of *an agent* $i \in I$ *as*

$$\frac{\partial \overline{q}}{\partial \tau_{e_i}} = \left\{ \left\{ \frac{\partial \overline{q}(\xi,\, k,\, \tau)}{\partial \tau_{e_i}(\xi',\, l)} \right\}_{(k,\, l) \in K \times L} \right\}_{(\xi',\, \xi) \in ET \times ET} \in \mathbb{R}^{|ET \times ET \times K \times L|}.$$

Set also

$$\frac{\partial \overline{q}(\xi',\, \tau_d)}{\partial \tau_{e_i}(\xi)} = \left\{ \frac{\partial \overline{q}(\xi,\, k,\, \tau)}{\partial \tau_{e_i}(\xi',\, l)} \right\}_{(k,\, l,) \in K \times L} \in \mathbb{R}^{|K \times L|}.$$

Clearly,

$$\frac{\partial \overline{q}}{\partial \tau_{e_i}} = \left\{ \frac{\partial \overline{q}(\xi',\, \tau_d)}{\partial \tau_{e_i}(\xi)} \right\}_{(\xi',\, \xi) \in ET \times ET}.$$

DEFINITION: *Define the matrix of partial derivatives of node prices* $\overline{\pi} \in \mathbb{R}^{|ET\times L|}$ *with respect to the endowment tax* $\tau_{e_i} \in [0,\ 1]^{|ET\times L|}$ of *an agent* $i \in I$ *as*

$$\frac{\partial\overline{\pi}}{\partial\tau_{e_i}} = \left\{ \left\{ \frac{\partial\overline{\pi}(\xi',\, l_1,\, \tau)}{\partial\tau_{e_i}(\xi,\, l_2)} \right\}_{(l_1,\, l_2)\in L\times L} \right\}_{(\xi',\, \xi)\in ET\times ET} \in \mathbb{R}^{|ET\times ET\times L\times L|}.$$

Set also

$$\frac{\partial\overline{\pi}(\xi',\, \tau)}{\partial\tau_{e_i}(\xi)} = \left\{ \frac{\partial\overline{\pi}(\xi',\, l_1,\, \tau)}{\partial\tau_{e_i}(\xi,\, l_2)} \right\}_{(l_1,\, l_2)\in L\times L} \in \mathbb{R}^{|L\times L|}.$$

Clearly,

$$\frac{\partial\overline{\pi}}{\partial\tau_{e_i}} = \left\{ \frac{\partial\overline{\pi}(\xi',\, \tau)}{\partial\tau_{e_i}(\xi)} \right\}_{(\xi',\, \xi)\in ET\times ET}.$$

DEFINITION: *Define the matrix of partial derivatives of the endowment tax* $\tau_{e_i} \in [0,\ 1]^{|ET\times L|}$ of *an agent* $i \in I$ *with respect to the endowment tax* $\tau_{e_i} \in [0,\ 1]^{|ET\times L|}$ of *an agent* $i \in I$ *itself as*

$$\frac{\partial\tau_{e_i}}{\partial\tau_{e_i}} = \left\{ \left\{ \frac{\partial\tau_{e_i}(\xi',\ l_1)}{\partial\tau_{e_i}(\xi,\ l_2)} \right\}_{(l_1,\, l_2)\in L\times L} \right\}_{(\xi',\, \xi)\in ET\times ET} \in \mathbb{R}^{|ET\times ET\times L\times L|}.$$

Set also

$$\frac{\partial\tau_{e_i}(\xi',\, \tau)}{\partial\tau_{e_i}(\xi)} = \left\{ \frac{\partial\tau_{e_i}(\xi',\ l_1)}{\partial\tau_{e_i}(\xi,\ l_2)} \right\}_{(l_1,\, l_2)\in L\times L} \in \mathbb{R}^{|L\times L|}.$$

Clearly,

$$\frac{\partial\tau_{e_i}}{\partial\tau_{e_i}} = \left\{ \frac{\partial\tau_{e_i}(\xi')}{\partial\tau_{e_i}(\xi)} \right\}_{(\xi',\, \xi)\in ET\times ET}.$$

We will assume for the rest of the paper

$$sign\left[\frac{\partial\overline{c}_i}{\partial\tau_d}\right] = sign\left[\frac{\partial\overline{c}_j}{\partial\tau_d}\right]\ \forall (i,\, j) \in I\times I$$

and

$$sign\left[\frac{\partial\overline{c}_i}{\partial\tau_{e_i}}\right] = sign\left[\frac{\partial\overline{c}_j}{\partial\tau_{e_j}}\right]\ \forall (i,\, j) \in I\times I.$$

2.2. Comparative Statics of FM Equilibria with Respect to the Dividend Tax τ_d

Let us first analyze how a change in the current dividend tax rate $\tau_d(\xi) \in \mathbb{R}^{|K|}$ will affect current equilibrium asset prices $\overline{q}(\xi,\ \tau) \in \mathbb{R}^{|K|}$. Since a change in $\tau_d(\xi)$ might affect various node prices $\overline{\pi}(\xi',\ \tau)$ and after-tax dividends $(1-\tau_d(\xi')) \cdot d(\xi',\ \tau_d)$, $\xi' \in ET^+(\xi)$ differently, the net effect of $\tau_d(\xi)$ on $\overline{q}(\xi,\ \tau)$ is ambiguous. We need to impose additional assumptions to remove this ambiguity. We will be able to derive economically intuitive comparative statics of $\overline{q}(\xi,\ \tau)$ with respect to $\tau_d(\xi)$ results without assuming either CRRA utility functions or identical agents. We will start our analysis with the following lemma:

LEMMA 2.2.1: *Let*

$$(\{(\overline{c}_i(\tau),\ \overline{z}_i(\tau))\}_{i\in I},\ (\overline{p}(\tau),\ \overline{q}(\tau))) \in \left(\mathbb{R}_+^{|ET\times L\times I|} \times \mathcal{Z}^{|I|}\right) \times \left(\mathbb{R}_+^{|ET\times L|} \times Q\ \right)$$

be an FM equilibrium in which markets are complete for the FM economy $\mathcal{E}(ET,$ $(1-\tau_e)\cdot e(\tau_e),\ \succeq,\ A(\tau_d))$ with stochastic taxation

$$\tau = (\tau_e,\ \tau_d) \in [0,1[^{|ET\times L\times I|} \times [0,\ 1]^{|ET\times K|},$$

where agents' preferences $\succeq_i$ on $\mathbb{R}_+^{|\mathbb{ET}\times\mathbb{L}|}\times\mathbb{R}_+^{|\mathbb{ET}\times\mathbb{L}|}$ are given by the utility function

$$U_i(c_i,\ G) = \sum_{(\xi,\ l)\in ET\times L} \Pr(\xi)\cdot b_i^{T(\xi)} \cdot [u_i(c_i(\xi,\ l)) + v_i(G(\xi,\ l))],$$

where $u_i \in C^2$ such that $u_i'(\cdot) > 0$ and $u_i"(\cdot) < 0\ \forall i \in I$. Assume further

$$\exists \tfrac{\partial\overline{c}_i}{\partial\tau_d} = \left\{\frac{\partial\overline{c}_i(\xi',\ \tau)}{\partial\tau_d(\overline{\xi})}\right\}_{(\xi',\ \overline{\xi})\in ET\times ET},\ s.t.\ \frac{\partial\overline{c}_i(\xi',\ \tau)}{\partial\tau_d(\overline{\xi})} = 0\ \forall\overline{\xi} \in ET \setminus ET(\xi').$$

Let

$$\overline{\pi}(\tau) = \{\overline{\pi}(\xi',\ \tau)\}_{\xi'\in ET} \in \mathbb{R}_{++}^{|ET|}$$

be the unique normalized price vector such that

$$\overline{\pi}(\tau)\cdot W(q(\tau),\ (1-\tau_d)\cdot d(\tau_d)) = 0.$$

Then

$$\boxed{\overline{\pi}(\xi',\,\tau)=b_i^{T(\xi')}\cdot\frac{u_i'(\overline{c}_i(\xi',\,1,\,\tau))}{u_i'(\overline{c}_i(\xi,\,1,\,\tau))}\cdot\Pr(\xi')\;\forall(\xi',\,i)\in ET^{+}(\xi)\times I} \tag{1}$$

and

$$\boxed{\frac{\partial\overline{\pi}(\xi',\,\tau)}{\partial\tau_d\big(\overline{\xi}\big)}=\overline{\pi}(\xi',\tau)\cdot\left[rr_i(\overline{c}_i(\xi,\;1,\;\tau))\cdot g_{\overline{c}_i(\xi,\,1,\,\tau)}-rr_i(\overline{c}_i(\xi',\;1,\;\tau))\cdot g_{\overline{c}_i(\xi',\,1,\,\tau)}\right]\;\forall\overline{\xi}\in ET,} \tag{2}$$

where ξ *is the initial node of the event tree* ET*,*

$$rr_i(c)=-\left[\frac{u_i^{"}(c)\cdot c}{u_i'(c)}\right]$$

is the coefficient of relative risk aversion of an agent $i\in I$ *and*

$$g_{\overline{c}_i(\xi',\,1,\,\tau)}=\frac{1}{\overline{c}_i(\xi',\,1,\,\tau)}\cdot\frac{\partial\overline{c}_i(\xi',\,1,\,\tau)}{\partial\tau_d\big(\overline{\xi}\big)}\;\forall\xi'\in ET.$$

PROOF: See Appendix.
THEOREM 2.2.2: *Let*

$$\left(\{(\overline{c}_i(\tau),\;\overline{z}_i(\tau))\}_{i\in I}\,,\,(\overline{p}(\tau),\;\overline{q}(\tau))\right)\in\left(\mathbb{R}_+^{|ET\times L\times I|}\times\mathcal{Z}^{|I|}\right)\times\left(\mathbb{R}_+^{|ET\times L|}\times Q\;\right)$$

be an FM equilibrium in which markets are complete for the FM economy $\mathcal{E}(ET,$ $(1-\tau_e)\cdot e(\tau_e),\,\succeq,\,A(\tau_d))$ *with stochastic taxation*

$$\tau=(\tau_e,\;\tau_d)\in[0,1[^{|ET\times L\times I|}\times[0,\;1]^{|ET\times K|}.$$

Suppose in addition to the assumptions of Lemma 2.2.1. we also assume that

$$\exists\frac{\partial d}{\partial\tau_d}=\left\{\frac{\partial d(\xi',\,\tau_d)}{\partial\tau_d\big(\overline{\xi}\big)}\right\}_{(\xi',\,\overline{\xi})\in ET\times ET},\;s.t.\;\frac{\partial d(\xi',\,\tau_d)}{\partial\tau_d\big(\overline{\xi}\big)}=0\;\forall\overline{\xi}\in ET\setminus ET\,(\xi')$$

and

$$\exists\frac{\partial\tau_d}{\partial\tau_d}=\left\{\left\{\frac{\partial\tau_d(\xi',\,k_1)}{\partial\tau_d\big(\overline{\xi},\;k_2\big)}\right\}_{(k_1,\,k_2,)\in K\times K}\right\}_{(\xi',\,\xi)\in ET\times ET},\;s.t.\;\frac{\partial\tau_d(\xi',\,k_1)}{\partial\tau_d\big(\overline{\xi},\;k_2\big)}=0$$

$\forall \left((\xi', \ k_1), \ \left(\overline{\xi}, \ k_2\right) \right) \in [ET \times K] \times [[ET \times K] \setminus \{(\xi', \ k_1)\}]$.
Then

$$\boxed{\overline{q}\,(\xi, \ \tau) = \sum_{\xi' \in ET^+(\xi)} b_i^{T(\xi')} \cdot \frac{u_i'(\overline{c}_i(\xi', \, 1, \, \tau))}{u_i'(\overline{c}_i(\xi, \, 1, \, \tau))} \cdot \Pr(\xi') \cdot (1 - \tau_d\,(\xi')) \cdot d\,(\xi', \ \tau_d),} \tag{3}$$

$$\boxed{\frac{\partial \overline{q}(\xi, \, \tau)}{\partial \tau_d(\xi)} = rr_i(\overline{c}_i\,(\xi, \ 1, \ \tau)) \cdot g_{\overline{c}_i(\xi, \, 1, \, \tau)} \cdot \overline{q}\,(\xi, \ \tau)} \tag{4}$$

and

$$\boxed{sign \left[\frac{\partial \overline{q}(\xi, \, \tau)}{\partial \tau_d(\xi)} \right] = sign \left[\frac{\partial \overline{c}_i(\xi, \, 1, \, \tau)}{\partial \tau_d(\xi)} \right] \forall\,(\xi, \ i) \in ET \times I,}$$

where ξ *is the initial node of the event tree* ET,

$$rr_i(c) = -\left[\frac{u_i''(c) \cdot c}{u_i'(c)} \right]$$

is the coefficient of relative risk aversion of an agent $i \in I$,

$$g_{\overline{c}_i(\xi, \, 1, \, \tau)} = \frac{1}{\overline{c}_i(\xi, \, 1, \, \tau)} \cdot \frac{\partial \overline{c}_i(\xi, \, 1, \, \tau)}{\partial \tau_d(\xi)}.$$

PROOF: See Appendix.

The economic interpretation of the above result is as follows. Note first that an increase in $\tau_d\,(\xi)$ affects $\overline{q}\,(\xi, \ \tau)$ only through stochastic discount factors $\overline{\pi}(\xi', \ \tau)$, $\xi' \in ET^+(\xi)$. Although the sign of a derivative of current equilibrium asset prices $\overline{q}\,(\xi, \ \tau)$ with respect to current dividend tax rates $\tau_d\,(\xi)$ may be ambiguous, it is always the same as that of the derivative of the current equilibrium consumption $\overline{c}_i\,(\xi, \ 1, \ \tau)$ of the numeraire good 1 with respect to current dividend tax rates $\tau_d\,(\xi)$. Depending on the sign of $\frac{\partial \overline{c}_i(\xi, \, 1, \, \tau)}{\partial \tau_d(\xi)}$, an increase in $\tau_d\,(\xi)$ might be reducing or boosting $\overline{q}\,(\xi, \ \tau)$. Therefore, we have two cases to consider here:

Suppose first $\frac{\partial \overline{c}_i(\xi, \, 1, \, \tau)}{\partial \tau_d(\xi)} \leqslant 0$, i.e., the numeraire good 1 is a normal good. Then an increase in $\tau_d\,(\xi)$ reduces $\overline{\pi}(\xi', \ \tau)$, thus decreasing today's price of the future consumption $\overline{c}_i\,(\xi', \ 1, \ \tau)$, $\xi' \in ET^+(\xi)$ of the numeraire good 1. Since financial assets represent claims on future consumption, the increase in $\tau_d\,(\xi)$ reduces today's asset prices $\overline{q}\,(\xi, \ \tau)$.

Suppose now $\frac{\partial \overline{c}_i(\xi, \, 1, \, \tau)}{\partial \tau_d(\xi)} > 0$, i.e., the numeraire good 1 is an inferior good. Then an increase in $\tau_d\,(\xi)$ boosts $\overline{\pi}(\xi', \ \tau)$, thus increasing today's price of the future consumption $\overline{c}_i\,(\xi', \ 1, \ \tau)$, $\xi' \in ET^+(\xi)$ of the numeraire good 1. Since financial assets represent claims on future consumption, the increase in $\tau_d(\xi)$ boosts today's asset prices $\overline{q}(\xi,\ \tau)$.

While inferior goods have been demonstrated to exist, it is generally believed that they are rare: at any given time and price level, the demand for the vast majority of goods moves in the intuitive direction with respect to changes in after-tax income. So it is reasonable to assume that the numeraire good 1 is a normal good. Therefore, it is natural to conclude that under reasonable assumptions, an increase in the current dividend tax rate reduces current asset prices.

COROLLARY 2.2.3: *Suppose assumptions of the above Theorem 2.2.2. hold. Assume further that all agents are identical and exhibit CRRA, i.e.,*

$$rr_i(c) = -\left[\frac{u_i^{"}(c)\cdot c}{u_i^{\prime}(c)}\right] = a\ \forall i \in I.$$

Assets' dividends are taxed identically, i.e.,

$$\tau_d(\xi,\ k) = \tau_d(\xi)\ \forall\,(\xi,\ k) \in ET \times K.$$

In addition, agents have zero initial endowments, i.e.,

$$e_i(\xi,\ 1,\ \tau) = 0\ \forall\,(i,\ \xi) \in I \times ET^{10}$$

and

$$\frac{\partial d}{\partial \tau_d} = \left\{\frac{\partial d(\xi',\ \tau_d)}{\partial \tau_d(\overline{\xi})}\right\}_{(\xi',\ \overline{\xi}) \in ET \times ET} = 0.$$

Then

$$\boxed{E_{q(\xi,\ \tau),\ 1-\tau_d(\xi)} = a,}$$

where

$$E_{q(\xi,\ \tau),\ 1-\tau_d(\xi)} = \frac{\frac{1}{\overline{q}(\xi,\ \tau)}\frac{\partial \overline{q}(\xi,\ \tau)}{\partial \tau_d(\xi)}}{\frac{1}{(1-\tau_d(\xi))}\frac{\partial(1-\tau_d(\xi))}{\partial \tau_d(\xi)}}\ \forall \xi \in ET$$

is the elasticity of asset prices $q(\xi,\ \tau)$ with respect to the economic freedom $1 - \tau_d(\xi)$ at a node $\xi \in ET$.

PROOF: See Appendix.

The economic interpretation of the above result is as follows. Since all agents are identical, the total supply of assets is given by $\overline{z} = \{\overline{z}(k)\}_{k\in K} \in \mathbb{R}_+^{|K|}$. Therefore, $\mathcal{E}(ET,\ (1-\tau_e)\cdot e(\tau_e),\ \succeq,\ A(\tau_d))$ is a production economy. Thus,

[10] We can obtain a similar result without assuming zero initial endowments. It is sufficient to assume $\tau_d(\xi) = \tau_e(\xi)\ \forall \xi \in ET$ instead.

$$\sum_{k\in K}\overline{z}(k)\cdot d(\xi,\,k,\,\tau_d)$$

can be interpreted as the country's total GDP at node $\xi \in ET$ and

$$(1-\tau_d(\xi)) = \frac{\sum_{k\in K}\overline{z}(k)\cdot(1-\tau_d(\xi))\cdot d(\xi,\,k,\,\tau_d)}{\sum_{k\in K}\overline{z}(k)\cdot d(\xi,\,k,\,\tau_d)}$$

can be interpreted as a percentage of the economy's total GDP consumed by the private sector at a node $\xi \in ET$. Hence, $(1-\tau_d(\xi))$ can be interpreted as the economy's level of economic freedom at a node $\xi \in ET$. Numerically, Magin (2015a) estimated the coefficient of agents' relative risk aversion, $a = 3.76$. So on average, for S&P 500 stocks, a 1% increase of the economy's economic freedom generates a 3.76% increase in share prices.

COROLLARY 2.2.4: *Suppose assumptions of Theorem 2.2.2. hold. Then*

$$\boxed{sign\left[\frac{\partial R(\xi',\,\tau)}{\partial \tau_d(\xi)}\right] = sign\left[\frac{\partial ATR(\xi',\,\tau)}{\partial \tau_d(\xi)}\right] = -sign\left[\frac{\partial q(\xi,\,\tau)}{\partial \tau(\xi)}\right]\ \forall\,(\xi,\ \xi') \in ET\ \times \xi^{+},}$$

where

$$R(\xi',\,\tau) = \frac{q(\xi',\,\tau)+d(\xi',\,\tau_d)}{q(\xi,\,\tau)}$$

is the total before-tax rate of return of an asset at a node $\xi' \in \xi^{+}$ and

$$ATR(\xi',\,\tau) = \frac{q(\xi',\,\tau)+(1-\tau_d(\xi'))\cdot d(\xi',\,\tau_d)}{q(\xi,\,\tau)}$$

is the total after-tax rate of return of an asset at a node $\xi' \in \xi^{+}$.

PROOF: See Appendix.

Empirical findings of Sialm (2006), Sialm (2009) and Hong and Kacperczyk (2009) imply that stocks with heavier tax and regulatory burdens tend to compensate taxable investors by offering higher before-tax returns and equity premia. Corollary 2.2.4. demonstrates that this is not necessarily the case.

Let us now analyze how a change in a future $\tau_d\left(\overline{\xi}\right)$, $\overline{\xi} \in ET^{+}(\xi)$ stochastic dividend tax rate will affect current equilibrium asset prices $\overline{q}\,(\xi,\ \tau)$. Again, since a change in $\tau_d\left(\overline{\xi}\right)$ might affect various node prices $\overline{\pi}(\xi',\,\tau)$, $\xi' \in ET^{+}(\xi)$ and after-tax dividends $(1-\tau_d\,(\xi'))\cdot d\,(\xi',\ \tau_d)$, $\xi' \in ET^{+}(\xi)$ differently, the net effect of $\tau_d\left(\overline{\xi}\right)$ on $\overline{q}\,(\xi,\ \tau)$ is ambiguous. Unlike the comparative statics of $\overline{q}\,(\xi,\ \tau)$ with respect to $\tau_d\,(\xi)$, it does not appear to be possible to derive economically intuitive comparative statics of $\overline{q}\,(\xi,\ \tau)$ with respect to $\tau_d\left(\overline{\xi}\right)$ results without assuming either CRRA utility functions or identical agents.

Suppose agents exhibit CRRA but are not necessarily identical.

THEOREM 2.2.5: *Let*

$$\left(\left\{\left(\overline{c}_i(\tau),\ \overline{z}_i(\tau)\right)\right\}_{i\in I},\ \left(\overline{p}(\tau),\ \overline{q}(\tau)\right)\right)\in\left(\mathbb{R}_+^{|ET\times L\times I|}\times\mathcal{Z}^{|I|}\right)\times\left(\mathbb{R}_+^{|ET\times L|}\times Q\ \right)$$

be an FM equilibrium in which markets are complete for the FM economy $\mathcal{E}(ET,$ $(1-\tau_e)\cdot e(\tau_e),\ \succeq,\ A(\tau_d))$ *with stochastic taxation*

$$\tau=(\tau_e,\ \tau_d)\in[0,1[^{|ET\times L\times I|}\times[0,\ 1]^{|ET\times K|},$$

where agents' preferences $\succeq_i$ *on* $\mathbb{R}_+^{|E\mathbb{T}\times\mathbb{L}|}\times\mathbb{R}_+^{|E\mathbb{T}\times\mathbb{L}|}$ *are given by the utility function*

$$U_i(c_i,\,G)=\sum_{(\xi,\,l)\in ET\times L}\Pr(\xi)\cdot b_i^{T(\xi)}\cdot\left[u_i(c_i(\xi,\ l))+v_i(G(\xi,\ l))\right]\ \forall i\in I,$$

where u_i *is a CRRA utility function such that* $u_i(c)=\frac{c^{1-a_i}}{1-a_i}$.
Assume further that

$$sign\left[\frac{\partial\overline{c}_i}{\partial\tau_d}\right]=-1\ \forall i\in I,$$

$$\exists\frac{\partial d}{\partial\tau_d}=\left\{\frac{\partial d(\xi',\,\tau_d)}{\partial\tau_d\left(\overline{\xi}\right)}\right\}_{(\xi',\,\overline{\xi})\in ET\times ET},\ s.t.\ \frac{\partial d(\xi',\,\tau_d)}{\partial\tau_d\left(\overline{\xi}\right)}=0\ \forall\overline{\xi}\in ET\setminus ET\left(\xi'\right),$$

$$\exists\frac{\partial\tau_d}{\partial\tau_d}=\left\{\left\{\frac{\partial\tau_d(\xi',\,k_1)}{\partial\tau_d\left(\overline{\xi},\ k_2\right)}\right\}_{(k_1,\,k_2,)\in K\times K}\right\}_{(\xi',\,\xi)\in ET\times ET},\ s.t.\ \frac{\partial\tau_d(\xi',\,k_1)}{\partial\tau_d\left(\overline{\xi},\ k_2\right)}=0$$

$\forall\left((\xi',\ k_1),\ \left(\overline{\xi},\ k_2\right)\right)\in[ET\times K]\times[[ET\times K]\setminus\{(\xi',\ k_1)\}]$
and

$$\frac{\overline{c}_i(\xi',\,1,\,\tau)}{\overline{c}_i(\xi,\,1,\,\tau)}=\frac{\sum\limits_{i\in I}\left(1-\tau_{e_i}(\xi',\,1)\right)\cdot e_i(\xi',\,1,\,\tau_{e_i})+\sum\limits_{k\in K}\overline{z}(k)\cdot(1-\tau_d(\xi',\,k))\cdot d(\xi',\,k,\,\tau_d)}{\sum\limits_{i\in I}\left(1-\tau_{e_i}(\xi,\,1)\right)\cdot e_i(\xi,\,1,\,\tau_{e_i})+\sum\limits_{k\in K}\overline{z}(k)\cdot(1-\tau_d(\xi,\,k))\cdot d(\xi,\,k,\,\tau_d)}$$

$\forall\,(\xi,\ \xi',\ i)\in ET\times ET\times I$, *where* $\overline{z}(k)$ *is the total number of outstanding shares of an asset* $k\in K$. *Let* ξ *be the initial node of the event tree* ET. *Fix* $\overline{\xi}\in ET^+(\xi)$. *Then*

$$\boxed{sign\left[\frac{\partial\overline{q}(\xi,\,\tau)}{\partial\tau_d\left(\overline{\xi}\right)}\right] = sign\left[a_i - B_i(\overline{\xi})\right],}$$

where

$$\frac{\partial\overline{q}(\xi,\,\tau)}{\partial\tau_d\left(\overline{\xi}\right)} = \left\{\frac{\partial\overline{q}(\xi,\,k_1,\,\tau)}{\partial\tau_d\left(\overline{\xi},\,k_2\right)}\right\}_{(k_1,\,k_2)\in K\times K} \in \mathbb{R}^{|K\times K|},$$

$$\boxed{B_i(\overline{\xi}) = \frac{d\left(\overline{\xi},\,\tau_d\right) - \frac{\partial d\left(\overline{\xi},\;\tau_d\right)}{\partial\tau_d\left(\overline{\xi}\right)}\cdot\left(1-\tau_d\left(\overline{\xi}\right)\right)}{-g_{\overline{c}_i(\overline{\xi},\,1,\,\tau)}\cdot\left(1-\tau_d\left(\overline{\xi}\right)\right)\cdot d\left(\overline{\xi},\,\tau_d\right)}\;\forall\left(\overline{\xi},\;i\right)\in ET^+(\xi)\times I.}$$

PROOF: See Appendix.

The economic interpretation of the above result is as follows. Fix $\overline{\xi} \in ET^+(\xi)$. Note first that an increase in $\tau_d\left(\overline{\xi}\right)$ affects $\overline{q}\,(\xi,\;\tau)$ through both stochastic discount factors $\overline{\pi}(\xi',\,\tau)$ and after-tax dividends $(1-\tau_d\,(\xi'))\cdot d\,(\xi',\;\tau_d)$, $\xi' \in ET^+(\xi)$. Under the assumptions of the theorem, an increase in $\tau_d\left(\overline{\xi}\right)$ generates two effects which are working in opposite directions. On the one hand, an increase in $\tau_d\left(\overline{\xi}\right)$ boosts the stochastic discount factor $\overline{\pi}(\overline{\xi},\,\tau)$, thus boosting asset prices $\overline{q}\,(\xi,\;\tau)$. On the other hand, an increase in $\tau_d\left(\overline{\xi}\right)$ reduces after-tax dividends $\left(1-\tau_d\left(\overline{\xi}\right)\right)\cdot d\left(\overline{\xi},\;\tau_d\right)$, thus reducing asset prices $\overline{q}\,(\xi,\;\tau)$. The net effect of the increase in $\tau_d\left(\overline{\xi}\right)$ on $\overline{q}\,(\xi,\;\tau)$ is ambiguous and is determined by the value of the coefficient of relative risk aversion a_i. We have three cases to consider here:

If $a_i < B_i(\overline{\xi})\;\forall i \in I$, i.e., the coefficient of relative risk aversion a_i is low $\forall i \in I$, then an increase in $\tau_d\left(\overline{\xi}\right)$ generates such a strong boosting effect on the stochastic discount factor $\overline{\pi}(\overline{\xi},\;\tau)$ that it dominates the reducing effect it has on after-tax dividends $\left(1-\tau_d\left(\overline{\xi}\right)\right)\cdot d\left(\overline{\xi},\;\tau_d\right)$, thus boosting asset prices $\overline{q}\,(\xi,\;\tau)$.

If $a_i > B_i(\overline{\xi})\;\forall i \in I$, i.e., the coefficient of relative risk aversion a_i is high $\forall i \in I$, then an increase in $\tau_d\left(\overline{\xi}\right)$ generates such a weak boosting effect on the stochastic discount factor $\overline{\pi}(\overline{\xi},\,\tau)$ that it is dominated by the reducing effect it has on after-tax dividends $\left(1-\tau_d\left(\overline{\xi}\right)\right)\cdot d\left(\overline{\xi},\;\tau_d\right)$, thus reducing asset prices $\overline{q}\,(\xi,\;\tau)$.

Finally, if $a_i = B_i(\overline{\xi})\;\forall i \in I$, then an increase in $\tau_d\left(\overline{\xi}\right)$ generates such a boosting effect on the stochastic discount factor $\overline{\pi}(\overline{\xi},\,\tau)$ that it is completely canceled out by the reducing effect it has on after-tax dividends $\left(1-\tau_d\left(\overline{\xi}\right)\right)\cdot d\left(\overline{\xi},\;\tau_d\right)$, thus leaving asset prices $\overline{q}\,(\xi,\;\tau)$ unchanged.

Suppose agents are identical but do not necessarily exhibit CRRA.

THEOREM 2.2.6: *Let*

$$\left(\left\{\left(\overline{c}_i(\tau),\;\overline{z}_i(\tau)\right)\right\}_{i\in I},\,\left(\overline{p}(\tau),\;\overline{q}(\tau)\right)\right) \in \left(\mathbb{R}_+^{|ET\times L\times I|}\times\mathcal{Z}^{|I|}\right)\times\left(\mathbb{R}_+^{|ET\times L|}\times Q\;\right)$$

be an FM equilibrium in which markets are complete for the FM economy $\mathcal{E}(ET,$ $(1-\tau_e)\cdot e(\tau_e),\ \succeq,\ A(\tau_d))$ *with stochastic taxation*

$$\tau = (\tau_e,\ \tau_d) \in [0,1[^{|ET\times L\times I|} \times [0,\ 1]^{|ET\times K|},$$

where identical agents with preferences $\succeq_i$ *on* $\mathbb{R}_+^{|\mathbb{ET}\times\mathbb{L}|}\times\mathbb{R}_+^{|\mathbb{ET}\times\mathbb{L}|}$ *are given by the utility function*

$$U_i(c,\, G) = \sum_{(\xi,\, l)\in ET\times L} \Pr(\xi)\cdot b^{T(\xi)}\cdot[u(c(\xi,\ l)) + v(G(\xi,\ l))]\ \forall i\in I,$$

where $u\in C^2$ *such that* $u'(\cdot)>0$ *and* $u"(\cdot)<0$.
Assume further that

$$\exists\frac{\partial d}{\partial\tau_d} = \left\{\frac{\partial d(\xi',\, \tau_d)}{\partial\tau_d(\overline{\xi})}\right\}_{(\xi',\, \overline{\xi})\in ET\times ET},\ s.t.$$

$$sign\left[\frac{\partial d(\xi',\, \tau_d)}{\partial\tau_d(\overline{\xi})}\right] = \begin{cases} -1\ for\ \overline{\xi}=\xi' \\ 0\ \forall\overline{\xi}\in ET\setminus ET\,(\xi') \end{cases}$$

$$\exists\frac{\partial\tau_d}{\partial\tau_d} = \left\{\left\{\frac{\partial\tau_d(\xi',\, k_1)}{\partial\tau_d(\overline{\xi},\ k_2)}\right\}_{(k_1,\, k_2,)\in K\times K}\right\}_{(\xi',\, \xi)\in ET\times ET}\quad s.t.\ \frac{\partial\tau_d(\xi',\, k_1)}{\partial\tau_d(\overline{\xi},\ k_2)} = 0$$

$\forall\left((\xi',\ k_1),\ (\overline{\xi},\ k_2)\right)\in[ET\times K]\times[[ET\times K]\setminus\{(\xi',\ k_1)\}]$. *Let* ξ *be the initial node of the event tree* ET. *Fix* $\overline{\xi}\in ET^+(\xi)$. *Then*

$$\boxed{sign\left[\frac{\partial\overline{q}(\xi,\, \tau)}{\partial\tau_d(\overline{\xi})}\right] = sign\left[\mathbf{rr}(\overline{\mathbf{c}}(\overline{\boldsymbol{\xi}},\, \mathbf{1},\, \boldsymbol{\tau})) - B(\overline{\xi})\right],}$$

where

$$\frac{\partial\overline{q}(\xi,\, \tau)}{\partial\tau_d(\overline{\xi})} = \left\{\frac{\partial\overline{q}(\xi,\, k_1,\, \tau)}{\partial\tau_d(\overline{\xi},\, k_2)}\right\}_{(k_1,\, k_2)\in K\times K} \in\mathbb{R}^{|K\times K|},$$

$$\mathbf{rr}(\overline{\mathbf{c}}(\overline{\boldsymbol{\xi}},\, \mathbf{1},\, \boldsymbol{\tau})) = \left\{rr\left(\overline{c}(\overline{\xi},\ 1,\ \tau)\right)\right\}_{(k_1,\, k_2)\in K\times K}\in\mathbb{R}^{|K\times K|},$$

$$rr(c) = -\left[\frac{u"(c)\cdot c}{u'(c)}\right]$$

is the coefficient of agents' relative risk aversion and

$$\boxed{B(\overline{\xi}) = \frac{d(\overline{\xi},\, \tau_d) - \frac{\partial d(\overline{\xi},\, \tau_d)}{\partial\tau_d(\overline{\xi})}\cdot\left(1-\tau_d(\overline{\xi})\right)}{-g_{\overline{c}(\overline{\xi},\, 1,\, \tau)}\cdot\left(1-\tau_d(\overline{\xi})\right)\cdot d(\overline{\xi},\, \tau_d)}\ \forall\left(\xi,\ \overline{\xi}\right)\in ET\times ET^+(\xi).}$$

PROOF: See Appendix.

COROLLARY 2.2.7: *Suppose assumptions of the above Theorem 2.2.6. hold, except that now*

$$\frac{\partial d}{\partial \tau_d} = \left\{ \frac{\partial d(\xi',\, \tau_d)}{\partial \tau_d\left(\overline{\xi}\right)} \right\}_{(\xi',\, \overline{\xi}) \in ET \times ET} = 0.$$

Assume further that identical agents have zero initial endowments, i.e.,

$$e_i(\xi,\, 1,\, \tau) = 0 \;\forall\, (i,\; \xi) \in I \times ET$$

and all assets' dividends are taxed identically, i.e.,

$$\tau_d(\xi,\, k) = \tau_d(\xi) \;\forall\, (\xi,\; k) \in ET \times K.$$

Then

$$\boxed{sign \left[\frac{\partial \overline{q}(\xi,\, \tau)}{\partial \tau_d\left(\overline{\xi}\right)} \right] = sign \left[rr(\overline{c}(\overline{\xi},\; 1,\; \tau)) - 1 \right] \;\forall \overline{\xi} \in ET^{+}(\xi).}$$

PROOF: See Appendix.

2.3. Comparative Statics of FM Equilibria with Respect to the Endowment Tax τ_{e_i}

The analysis of how a change in the stochastic endowment tax τ_{e_i} of an agent $i \in I$ will affect equilibrium asset prices $\overline{q}\,(\tau)$ is much easier than the effects of τ_d on $\overline{q}\,(\tau)$, since, under reasonable assumptions, increases in τ_{e_i} affect $\overline{q}\,(\tau)$ only through stochastic discount factors $\overline{\pi}(\tau) \in \mathbb{R}^{|ET|}_{++}$.

Let us first analyze how a change in the current $\tau_{e_i}\,(\xi)$ endowment tax rate of an agent $i \in I$ will affect current equilibrium asset prices $\overline{q}\,(\xi,\; \tau)$. We will start with the following lemma:

LEMMA 2.3.1: *Let*

$$\left(\{(\overline{c}_i(\tau),\ \overline{z}_i(\tau))\}_{i\in I},\ (\overline{p}(\tau),\ \overline{q}(\tau))\right) \in \left(\mathbb{R}_+^{|ET\times L\times I|} \times \mathcal{Z}^{|I|}\right) \times \left(\mathbb{R}_+^{|ET\times L|} \times Q\ \right)$$

be an FM equilibrium in which markets are complete for the FM economy $\mathcal{E}(ET,$ $(1-\tau_e)\cdot e(\tau_e),\ \succeq,\ A(\tau_d))$ *with stochastic taxation*

$$\tau = (\tau_e,\ \tau_d) \in [0, 1[^{|ET\times L\times I|} \times [0,\ 1]^{|ET\times K|},$$

where agents' preferences $\succeq_i$ *on* $\mathbb{R}_+^{|E\mathbb{T}\times\mathbb{L}|}\times\mathbb{R}_+^{|E\mathbb{T}\times\mathbb{L}|}$ *are given by the utility function*

$$U_i(c_i,\, G) = \sum_{(\xi,\ l)\in ET\times L} \Pr(\xi)\cdot b_i^{T(\xi)}\cdot[u_i(c_i(\xi,\ l)) + v_i(G(\xi,\ l))],$$

where $u_i \in C^2$ *such that* $u_i'(\cdot) > 0$ *and* $u_i''(\cdot) < 0\ \forall i \in I$*. Assume further*

$$\exists \frac{\partial \overline{c}}{\partial \tau_{e_i}} = \left\{\frac{\partial \overline{c}_i(\xi',\ \tau)}{\partial \tau_{e_i}(\overline{\xi})}\right\}_{(\xi',\ \overline{\xi})\in ET\times ET},\ s.t.\ \frac{\partial \overline{c}_i(\xi',\ \tau)}{\partial \tau_{e_i}(\overline{\xi})} = 0\ \forall\overline{\xi} \in ET\setminus ET\,(\xi')\,.$$

Let

$$\overline{\pi}(\tau) = \{\overline{\pi}(\xi',\ \tau)\}_{\xi'\in ET} \in \mathbb{R}_{++}^{|ET|}$$

be the unique normalized price vector such that

$$\overline{\pi}(\tau)\cdot W(\overline{q}(\tau),\,(1-\tau_d)\cdot d(\tau_d)) = 0.$$

Then

$$\boxed{\overline{\pi}(\xi',\,\tau) = b_i^{T(\xi')}\cdot\frac{u_i'(\overline{c}_i(\xi',\,1,\,\tau))}{u_i'(\overline{c}_i(\xi,\,1,\,\tau))}\cdot\Pr(\xi')\ \forall(\xi',\,i)\in ET^+(\xi)\times I} \tag{1}$$

and

$$\boxed{\frac{\partial\overline{\pi}(\xi',\,\tau)}{\partial\tau_{e_i}(\overline{\xi})} = \overline{\pi}(\xi',\tau)\cdot\left[rr_i(\overline{c}_i(\xi,\ 1,\ \tau))\cdot g_{\overline{c}_i(\xi,\,1,\,\tau)} - rr_i(\overline{c}_i(\xi',\ 1,\ \tau))\cdot g_{\overline{c}_i(\xi',\,1,\,\tau)}\right]\ \forall\overline{\xi}\in ET,} \tag{5}$$

where ξ *be the initial node of the event tree* ET

$$rr_i(c) = -\left[\frac{u_i''(c)\cdot c}{u_i'(c)}\right]$$

is the coefficient of relative risk aversion of an agent $i \in I$ and

$$g_{\overline{c}_i(\xi',\,1,\,\tau)} = \frac{1}{\overline{c}_i(\xi',\,1,\,\tau)} \cdot \frac{\partial \overline{c}_i(\xi',\,1,\,\tau)}{\partial \tau_{e_i}(\overline{\xi})}\ \forall \xi' \in ET.$$

PROOF: It is similar to the proof of Lemma 2.2.1. ■

THEOREM 2.3.2: *Let*

$$\left(\{(\overline{c}_i(\tau),\ \overline{z}_i(\tau))\}_{i\in I},\ (\overline{p}(\tau),\ \overline{q}(\tau))\right) \in \left(\mathbb{R}_+^{|ET\times L\times I|} \times \mathcal{Z}^{|I|}\right) \times \left(\mathbb{R}_+^{|ET\times L|} \times Q\right)$$

be an FM equilibrium in which markets are complete for the FM economy $\mathcal{E}(ET,$ $(1-\tau_e)\cdot e(\tau_e),\ \succeq,\ A(\tau_d))$ with stochastic taxation

$$\tau = (\tau_e,\ \tau_d) \in [0,1[^{|ET\times L\times I|} \times [0,\ 1]^{|ET\times K|}.$$

Suppose in addition to the assumptions of Lemma 2.3.1. we also assume that

$$\exists \frac{\partial d}{\partial \tau_{e_i}} = \left\{\frac{\partial d(\xi',\,\tau_d)}{\partial \tau_{e_i}(\overline{\xi})}\right\}_{(\xi',\,\overline{\xi})\in ET\times ET},\ s.t.\ \frac{\partial d(\xi',\,\tau_d)}{\partial \tau_{e_i}(\overline{\xi})} = 0\ \forall (\xi',\,\overline{\xi}) \in ET \times ET$$

and

$$\exists \frac{\partial \tau_d}{\partial \tau_{e_i}} = \left\{\left\{\frac{\partial \tau_d(\xi')}{\partial \tau_{e_i}(\overline{\xi})}\right\}\right\}_{(\xi',\,\overline{\xi})\in ET\times ET},\ s.t.\ \frac{\partial \tau_d(\xi')}{\partial \tau_{e_i}(\overline{\xi})} = 0\ \forall (\xi',\,\overline{\xi}) \in ET \times ET.$$

Then

$$\boxed{\overline{q}\,(\xi,\ \tau) = \sum_{\xi'\in ET^+(\xi)} b_i^{T(\xi')} \cdot \frac{u_i'(\overline{c}_i(\xi',\,1,\,\tau))}{u_i'(\overline{c}_i(\xi,\,1,\,\tau))} \cdot \Pr(\xi') \cdot (1-\tau_d\,(\xi')) \cdot d\,(\xi',\ \tau_d),} \qquad (3)$$

$$\boxed{\frac{\partial \overline{q}(\xi,\,\tau)}{\partial \tau_{e_i}(\xi)} = rr_i(\overline{c}_i(\xi,\,1,\,\tau)) \cdot g_{\overline{c}_i(\xi,\,1,\,\tau)} \cdot \overline{q}\,(\xi,\ \tau)}$$

and

$$\boxed{sign\ \boxed{\frac{\partial \overline{q}(\xi,\,\tau)}{\partial \tau_{e_i}(\xi)}} = sign\ \boxed{\frac{\partial \overline{c}_i(\xi,\,1,\,\tau)}{\partial \tau_{e_i}(\xi)}}\ \forall\,(\xi,\ i) \in ET \times I,}$$

where ξ is the initial node of the event tree ET,

$$rr_i(c) = -\left[\frac{u_i^{"}(c) \cdot c}{u_i'(c)}\right]$$

is the coefficient of relative risk aversion of an agent $i \in I$,

$$g_{\overline{c}_i(\xi,\ 1,\ \tau)} = \frac{1}{\overline{c}_i(\xi,\ 1,\ \tau)} \cdot \frac{\partial \overline{c}_i(\xi,\ 1,\ \tau)}{\partial \tau_{e_i}(\xi)}.$$

PROOF: It is similar to the proof of Theorem 2.2.2. ∎

The economic interpretation of the above result is as follows. Note first that an increase in $\tau_{e_i}(\xi,\ l)$ affects $\overline{q}(\xi,\ \tau)$ only through stochastic discount factors $\overline{\pi}(\xi',\tau)$, $\xi' \in ET^{+}(\xi)$. Although the sign of a derivative of current equilibrium asset prices $\overline{q}(\xi,\ \tau)$ with respect to current endowment tax rates $\tau_{e_i}(\xi)$ may be ambiguous, it is always the same as that of the derivative of the current equilibrium consumption $\overline{c}_i(\xi,\ 1,\ \tau)$ of the numeraire good 1 with respect to current endowment tax rates $\tau_{e_i}(\xi)$. Depending on the sign of $\frac{\partial \overline{c}_i(\xi,\ 1,\ \tau)}{\partial \tau_{e_i}(\xi)}$, an increase in $\tau_{e_i}(\xi)$ might be reducing or boosting $\overline{q}(\xi,\ \tau)$. Therefore, we have two cases to consider here:

Suppose first $\frac{\partial \overline{c}_i(\xi,\ 1,\ \tau)}{\partial \tau_{e_i}(\xi)} \leqslant 0$, i.e., the numeraire good 1 is a normal good. Then an increase in $\tau_{e_i}(\xi)$ reduces $\overline{\pi}(\xi',\ \tau)$, thus decreasing today's price of future consumption $\overline{c}_i(\xi',\ 1,\ \tau)$, $\xi' \in ET^{+}(\xi)$ of the numeraire good 1. Since financial assets represent claims on future consumption, the increase in $\tau_{e_i}(\xi)$ reduces today's asset prices $\overline{q}(\xi,\ \tau)$.

Suppose now $\frac{\partial \overline{c}_i(\xi,\ 1,\ \tau)}{\partial \tau_{e_i}(\xi)} > 0$, i.e., the numeraire good 1 is an inferior good. Then an increase in $\tau_{e_i}(\xi)$ boosts $\overline{\pi}(\xi',\tau)$, thus increasing today's price of future consumption $\overline{c}_i(\xi',\ 1,\ \tau)$, $\xi' \in ET^{+}(\xi)$ of the numeraire good 1. Since financial assets represent claims on future consumption, the increase in $\tau_{e_i}(\xi)$ boosts today's asset prices $\overline{q}(\xi,\ \tau)$.

While inferior goods have been demonstrated to exist, it is generally believed that they are rare: at any given time and price level, the demand for the vast majority of goods moves in the intuitive direction with respect to changes in after-tax income. So it is reasonable to assume that the numeraire good 1 is a normal good. Therefore, it is natural to conclude that under reasonable assumptions, an increase in the current endowment tax rate reduces current asset prices.

COROLLARY 2.3.3: *Suppose assumptions of the above Theorem 2.3.2. hold. Assume further that all agents are identical and exhibit CRRA, i.e.,*

$$rr_i(c) = -\left[\frac{u_i^{"}(c) \cdot c}{u_i'(c)}\right] = a\ \forall i \in I.$$

Agents' endowments are taxed identically, i.e.,

$$\tau_{e_i}(\xi) = \tau_e(\xi)\ \forall\,(\xi,\ i) \in ET \times I.$$

In addition, assets pay zero dividends, i.e.,

$$d(\xi,\ k,\ \tau_d) = 0\ \forall\,(\xi,\ k) \in ET \times K^{11}$$

and

$$\frac{\partial e}{\partial \tau_e} = \left\{ \frac{\partial \overline{c}_i(\xi',\ \tau)}{\partial \tau_{e_i}\left(\overline{\xi}\right)} \right\}_{(\xi',\ \overline{\xi}) \in ET \times ET} = 0.$$

Then

$$\boxed{E_{q(\xi,\ \tau),\ 1-\tau_e(\xi)} = a,}$$

where

$$E_{q(\xi,\ \tau),\ 1-\tau_e(\xi)} = \frac{\frac{1}{\overline{q}(\xi,\ \tau)} \frac{\partial \overline{q}(\xi,\ \tau)}{\partial \tau_e(\xi)}}{\frac{1}{(1-\tau_e(\xi))} \frac{\partial (1-\tau_e(\xi))}{\partial \tau_e(\xi)}}\ \forall \xi \in ET$$

is the elasticity of asset prices $q(\xi,\ \tau)$ with respect to the economic freedom $1 - \tau_e(\xi)$ at a node $\xi \in ET$.

PROOF: It is similar to the proof of Corollary 2.2.3. ■

Let us now analyze how a change in a future $\tau_{e_i}\left(\overline{\xi}\right)$, $\overline{\xi} \in ET^+(\xi)$ stochastic endowment tax rate of an agent $i \in I$ will affect current equilibrium asset prices $\overline{q}\,(\xi,\ \tau)$. Since a change in $\tau_{e_i}\left(\overline{\xi}\right)$ might affect various node prices $\overline{\pi}(\xi',\ \tau)$, $\xi' \in ET^+(\xi)$ differently, the net effect of $\tau_{e_i}\left(\overline{\xi}\right)$ on $\overline{q}\,(\xi,\ \tau)$ is ambiguous. Unlike the comparative statics of $\overline{q}\,(\xi,\ \tau)$ with respect to $\tau_{e_i}\,(\xi)$, however, it does not appear to be possible to derive economically intuitive comparative statics of $\overline{q}\,(\xi,\ \tau)$ with respect to $\tau_{e_i}\left(\overline{\xi}\right)$ results without either assuming CRRA utility functions or identical agents.

Suppose agents exhibit CRRA but are not necessarily identical.

THEOREM 2.3.4: *Let*

$$\left(\left\{\left(\overline{c}_i(\tau),\ \overline{z}_i(\tau)\right)\right\}_{i\in I},\ \left(\overline{p}(\tau),\ \overline{q}(\tau)\right)\right) \in \left(\mathbb{R}_+^{|ET\times L\times I|} \times \mathcal{Z}^{|I|}\right) \times \left(\mathbb{R}_+^{|ET\times L|} \times Q\ \right)$$

be an FM equilibrium in which markets are complete for the FM economy $\mathcal{E}(ET,$ $(1-\tau_e)\cdot e(\tau_e),\ \succeq,\ A(\tau_d))$ with stochastic taxation

[11] We can obtain a similar result without assuming that assets pay zero dividends. It is sufficient to assume $\tau_d(\xi) = \tau_e(\xi)\ \forall \xi \in ET$ instead.

$$\tau = (\tau_e,\ \tau_d) \in [0,1[^{|ET\times L\times I|} \times [0,\ 1]^{|ET\times K|},$$

where agents' preferences $\succeq_i$ *on* $\mathbb{R}_+^{|\mathbb{ET}\times\mathbb{L}|}\times\mathbb{R}_+^{|\mathbb{ET}\times\mathbb{L}|}$ *are given by the utility function*

$$U_i(c_i,\, G) = \sum_{(\xi,\, l)\in ET\times L} \Pr(\xi)\cdot b_i^{T(\xi)}\cdot[u_i(c_i(\xi,\ l)) + v_i(G(\xi,\ l))]\ \forall i\in I,$$

where u_i *is a CRRA utility function such that* $u_i(c) = \frac{c^{1-a_i}}{1-a_i}$.
Assume further that

$$\frac{\overline{c}_i(\xi',\,1,\,\tau)}{\overline{c}_i(\xi,\,1,\,\tau)} = \frac{\displaystyle\sum_{i\in I}\left(1-\tau_{e_i}(\xi',\,1)\right)\cdot e_i(\xi',\,1,\,\tau_{e_i})+\sum_{k\in K}\overline{z}(k)\cdot(1-\tau_d(\xi',\,k))\cdot d(\xi',\,k,\,\tau_d)}{\displaystyle\sum_{i\in I}\left(1-\tau_{e_i}(\xi,\,1)\right)\cdot e_i(\xi,\,1,\,\tau_{e_i})+\sum_{k\in K}\overline{z}(k)\cdot(1-\tau_d(\xi,\,k))\cdot d(\xi,\,k,\,\tau_d)}$$

$\forall\,(\xi,\ \xi',\ i) \in ET\times ET\times I$,

where $\overline{z}(k)$ *is the total number of outstanding shares of asset* $k\in K$. *Let* ξ *be the initial node of the event tree* ET. *Let*

$$\overline{\pi}(\tau) = \{\overline{\pi}(\xi',\ \tau)\}_{\xi\in ET} \in \mathbb{R}_{++}^{|ET|}$$

be the unique normalized price vector such that

$$\overline{\pi}(\tau)\cdot W(\overline{q}(\tau),\,(1-\tau_d)\cdot d(\tau_d)) = 0.$$

Fix $\overline{\xi}\in ET^+(\xi)$. *Then*

$$\boxed{\frac{\partial\overline{q}(\xi,\,\tau)}{\partial\tau_{e_i}\left(\overline{\xi}\right)} = -a_i\cdot\overline{\pi}(\overline{\xi},\tau)\cdot g_{\overline{c}_i(\overline{\xi},\,1,\,\tau)}\cdot\left(1-\tau_d\left(\overline{\xi}\right)\right)\cdot d\left(\overline{\xi},\ \tau_d\right)}$$

and

$$\boxed{sign\left[\frac{\partial\overline{q}(\xi,\,\tau)}{\partial\tau_{e_i}\left(\overline{\xi}\right)}\right] = -sign\left[\frac{\partial\overline{c}_i\left(\overline{\xi},\,1,\,\tau\right)}{\partial\tau_{e_i}\left(\overline{\xi}\right)}\right]\ \forall\overline{\xi}\in ET^+(\xi).}$$

PROOF: See Appendix.

The economic interpretation of the above result is as follows. Fix $\overline{\xi}\in ET^+(\xi)$. Note first that an increase in $\tau_{e_i}\left(\overline{\xi},\ l\right)$ affects $\overline{q}\,(\xi,\ \tau)$ only through stochastic discount factors $\overline{\pi}(\overline{\xi},\ \tau)$, $\xi'\in ET^+(\xi)$. Although the sign of a derivative of current equilibrium asset prices $\overline{q}\,(\xi,\ \tau)$ with respect to future endowment tax rates $\tau_{e_i}\left(\overline{\xi}\right)$ may be ambiguous, it is always the opposite of that of the derivative of equilibrium consumption $\overline{c}_i(\xi,\ 1,\ \tau)$ with respect to future endowment tax rates $\tau_{e_i}\left(\overline{\xi}\right)$. Depending on the sign of $\frac{\partial \overline{c}_i(\xi,\ 1,\ \tau)}{\partial \tau_{e_i}\left(\overline{\xi}\right)}$, an increase in $\tau_{e_i}\left(\overline{\xi}\right)$ might be reducing or boosting $\overline{q}(\xi,\ \tau)$. Therefore, we have two cases to consider here:

Suppose first $\frac{\partial \overline{c}_i\left(\overline{\xi},\ 1,\ \tau\right)}{\partial \tau_{e_i}\left(\overline{\xi},\ l\right)} \leqslant 0$, i.e., the numeraire good 1 is a normal good. Then an increase in $\tau_{e_i}\left(\overline{\xi},\ l\right)$ boosts $\overline{\pi}(\overline{\xi},\ \tau)$, thus increasing today's price of future consumption $\overline{c}_i\left(\overline{\xi},\ 1,\ \tau\right)$ of the numeraire good 1. Since financial assets represent claims on future consumption, the increase in $\tau_{e_i}\left(\overline{\xi},\ l\right)$ boosts today's asset prices $\overline{q}(\xi,\ \tau)$.

Suppose now $\frac{\partial \overline{c}_i\left(\overline{\xi},\ 1,\ \tau\right)}{\partial \tau_{e_i}\left(\overline{\xi},\ l\right)} > 0$, i.e., the numeraire good 1 is an inferior good. Then an increase in $\tau_{e_i}\left(\overline{\xi},\ l\right)$ reduces $\overline{\pi}(\overline{\xi},\ \tau)$, thus decreasing today's price of future consumption $\overline{c}_i\left(\overline{\xi},\ 1,\ \tau\right)$ of the numeraire good 1. Since financial assets represent claims on future consumption, the increase in $\tau_{e_i}\left(\overline{\xi},\ l\right)$ reduces today's asset prices $\overline{q}(\xi,\ \tau)$.

While inferior goods have been demonstrated to exist, it is generally believed that they are rare: at any given time and price level, the demand for the vast majority of goods moves in the intuitive direction with respect to changes in after-tax income. So it is reasonable to assume that the numeraire good 1 is a normal good. Therefore, it is natural to conclude that under reasonable assumptions, an increase in the future endowment tax rate reduces current asset prices.

Suppose agents are identical but do not necessarily exhibit CRRA.

THEOREM 2.3.5.: *Let*

$$\left(\{(\overline{c}_i(\tau),\ \overline{z}_i(\tau))\}_{i\in I},\ (\overline{p}(\tau),\ \overline{q}(\tau))\right) \in \left(\mathbb{R}_+^{|ET\times L\times I|} \times \mathcal{Z}^{|I|}\right) \times \left(\mathbb{R}_+^{|ET\times L|} \times Q\ \right)$$

be an FM equilibrium in which markets are complete for the FM economy $\mathcal{E}(ET,$ $(1-\tau_e)\cdot e(\tau_e),\ \succeq,\ A(\tau_d))$ with stochastic taxation

$$\tau = (\tau_e,\ \tau_d) \in [0,1[^{|ET\times L\times I|} \times [0,\ 1]^{|ET\times K|},$$

where identical agents with preferences $\succeq_i$ on $\mathbb{R}_+^{|\mathbb{ET}\times\mathbb{L}|} \times \mathbb{R}_+^{|\mathbb{ET}\times\mathbb{L}|}$ are given by the utility function

$$U_i(c,\ G) = \sum_{(\xi,\ l)\in ET\times L} \Pr(\xi)\cdot b^{T(\xi)}\cdot [u(c(\xi,\ l)) + v(G(\xi,\ l))]\ \forall i \in I,$$

where $u \in C^2$ such that $u'(\cdot) > 0$ and $u"(\cdot) < 0$. Let ξ be the initial node of the event tree ET. Let

$$\overline{\pi}(\tau) = \{\overline{\pi}(\xi',\ \tau)\}_{\xi\in ET} \in \mathbb{R}_{++}^{|ET|}$$

be the unique normalized price vector such that

$$\overline{\pi}(\tau) \cdot W(\overline{q}(\tau),\ (1-\tau_d) \cdot d(\tau_d)) = 0.$$

Fix $\overline{\xi} \in ET^{+}(\xi)$. *Then*

$$\boxed{\frac{\partial \overline{q}(\xi,\ \tau)}{\partial \tau_e(\overline{\xi})} = \overline{\pi}(\overline{\xi}, \tau) \cdot \left[rr(\overline{c}(\overline{\xi},\ 1,\ \tau))\right] \cdot \frac{e(\overline{\xi},\ 1,\ \tau_e)}{\overline{c}(\overline{\xi},\ 1,\ \tau)} \cdot \left(1-\tau_d\left(\overline{\xi}\right)\right) \cdot d\left(\overline{\xi},\ \tau_d\right) > 0.}$$

and

$$\boxed{sign\left[\frac{\partial \overline{q}(\xi,\ \tau)}{\partial \tau_e(\overline{\xi})}\right] = -sign\left[\frac{\partial \overline{c}\left(\overline{\xi},\ 1,\ \tau\right)}{\partial \tau_e\left(\overline{\xi}\right)}\right]\ \forall \overline{\xi} \in ET^{+}(\xi).}$$

PROOF: It is similar to the proof of Theorem 2.3.4. ■
The economic interpretation of this result is the same as for Theorem 2.3.4.

3. CONCLUSION

This paper studies comparative statics of FM equilibria in the finite horizon GEI model with respect to changes in stochastic tax rates imposed on agents' endowments and dividends. We show that under reasonable assumptions, without assuming CRRA and identical agents, an increase in the current dividend tax rate unambiguously reduces current asset prices. The paper also finds that there exists a bound $\overline{B}$ such that for a coefficient of relative risk aversion less than $\overline{B}$, an increase in a future dividend tax rate reduces current price of tradable assets. At the same time, for a coefficient of relative risk aversion greater than $\overline{B}$, an increase in a future dividend tax rate boosts current price of tradable assets. Finally, for a coefficient of relative risk aversion equal to $\overline{B}$, an increase in a future dividend tax rate leaves current consumption and current price of tradable assets unchanged. As a special case, under additional assumptions, $\overline{B}$ is equal to 1. Also, under reasonable assumptions, an increase in the current endowment tax rate reduces current asset prices, while an increase in a future endowment tax rate boosts current asset prices.

4. APPENDIX

Proofs for Comparative Statics of FM Equilibria with Respect to the Dividend Tax τ_d

PROOF OF LEMMA 2.2.1: Let ξ be the initial node of the event tree ET. Clearly,

$$\boxed{\overline{\pi}(\xi',\,\tau) = b_i^{T(\xi')} \cdot \frac{u_i'(\overline{c}_i(\xi',\,1,\,\tau))}{u_i'(\overline{c}_i(\xi,\,1,\,\tau))} \cdot \Pr(\xi')\ \forall(\xi',\,i) \in ET^{+}(\xi) \times I.} \tag{1}$$

Differentiating $\overline{\pi}(\xi',\,\tau)$ with respect to $\tau_d\left(\overline{\xi}\right)$ we obtain

$$\frac{\partial\overline{\pi}(\xi',\,\tau)}{\partial\tau_d\left(\overline{\xi}\right)} = b_i^{T(\xi')} \cdot \Pr(\xi') \cdot$$

$$\cdot \left[\frac{u_i''(\overline{c}_i(\xi',\,1,\,\tau)) \cdot \frac{\partial\overline{c}_i(\xi',\ 1,\ \tau)}{\partial\tau_d\left(\overline{\xi}\right)} \cdot u_i'(\overline{c}_i(\xi,\,1,\,\tau)) - u_i'(\overline{c}_i(\xi',\,1,\,\tau)) \cdot u_i''(\overline{c}_i(\xi,\,1,\,\tau)) \cdot \frac{\partial\overline{c}_i(\xi,\,1,\,\tau)}{\partial\tau_d\left(\overline{\xi}\right)}}{\left(u_i'(\overline{c}_i(\xi,\,1,\,\tau))\right)^2}\right] =$$

$$= b_i^{T(\xi')} \cdot \Pr(\xi') \cdot$$

$$\cdot \left[-\frac{u_i'(\overline{c}_i(\xi',\,1,\,\tau)) \cdot u_i''(\overline{c}_i(\xi,\,1,\,\tau))}{\left(u_i'(\overline{c}_i(\xi,\,1,\,\tau))\right)^2} \frac{\partial\overline{c}_i(\xi,\,1,\,\tau)}{\partial\tau_d\left(\overline{\xi}\right)} + \frac{u_i''(\overline{c}_i(\xi',\,1,\,\tau)) \cdot u_i'(\overline{c}_i(\xi,\,1,\,\tau))}{\left(u_i'(\overline{c}_i(\xi,\,1,\,\tau))\right)^2} \frac{\partial\overline{c}_i(\xi',\ 1,\ \tau)}{\partial\tau_d\left(\overline{\xi}\right)}\right] =$$

$$= b_i^{T(\xi')} \cdot \Pr(\xi') \cdot$$

$$\left[\frac{u_i'(\overline{c}_i(\xi',\,1,\,\tau))}{u_i'(\overline{c}_i(\xi,\,1,\,\tau))} \cdot [-\frac{u_i''(\overline{c}_i(\xi,\,1,\,\tau))}{\left(u_i'(\overline{c}_i(\xi,\,1,\,\tau))\right)}] \cdot \frac{\partial\overline{c}_i(\xi,\,1,\,\tau)}{\partial\tau_d\left(\overline{\xi}\right)} - \frac{u_i'(\overline{c}_i(\xi',\,1,\,\tau))}{u_i'(\overline{c}_i(\xi,\,1,\,\tau))} \cdot [-\frac{u_i''(\overline{c}_i(\xi',\,1,\,\tau))}{u_i'(\overline{c}_i(\xi',\,1,\,\tau))}] \cdot \frac{\partial\overline{c}_i(\xi',\ 1,\ \tau)}{\partial\tau_d\left(\overline{\xi}\right)}\right] =$$

$$= \left[b_i^{T(\xi')} \cdot \frac{u_i'(\overline{c}_i(\xi',\,1,\,\tau))}{u_i'(\overline{c}_i(\xi,\,1,\,\tau))} \cdot \Pr(\xi')\right] \cdot$$

$$\left[[-\frac{u_i''(\overline{c}_i(\xi,\,1,\,\tau))}{\left(u_i'(\overline{c}_i(\xi,\,1,\,\tau))\right)}] \cdot \frac{\partial\overline{c}_i(\xi,\,1,\,\tau)}{\partial\tau_d\left(\overline{\xi}\right)} - [-\frac{u_i''(\overline{c}_i(\xi',\,1,\,\tau))}{u_i'(\overline{c}_i(\xi',\,1,\,\tau))}] \cdot \frac{\partial\overline{c}_i(\xi',\ 1,\ \tau)}{\partial\tau_d\left(\overline{\xi}\right)}\right] =$$

$$= \overline{\pi}(\xi',\,\tau) \cdot \left[[-\frac{u_i''(\overline{c}_i(\xi,\,1,\,\tau))}{\left(u_i'(\overline{c}_i(\xi,\,1,\,\tau))\right)}] \cdot \frac{\partial\overline{c}_i(\xi,\,1,\,\tau)}{\partial\tau_d\left(\overline{\xi}\right)} - [-\frac{u_i''(\overline{c}_i(\xi',\,1,\,\tau))}{u_i'(\overline{c}_i(\xi',\,1,\,\tau))}] \cdot \frac{\partial\overline{c}_i(\xi',\ 1,\ \tau)}{\partial\tau_d\left(\overline{\xi}\right)}\right] =$$

$$= \overline{\pi}(\xi',\tau) \cdot \left[\underbrace{[-\frac{u_i''(\overline{c}_i(\xi,\ 1,\ \tau))}{u_i'(\overline{c}_i(\xi,\ 1,\ \tau))} \cdot \overline{c}_i(\xi,\ 1,\ \tau)]}_{rr_i(\overline{c}_i(\xi,\,1,\,\tau))} \cdot [\frac{1}{\overline{c}_i(\xi,\,1,\,\tau)} \frac{\partial\overline{c}_i(\xi,\,1,\,\tau)}{\partial\tau_d\left(\overline{\xi}\right)}] \right.$$

$$\left. - \underbrace{[-\frac{u_i''(\overline{c}_i(\xi',\ 1,\ \tau))}{u_i'(\overline{c}_i(\xi',\ 1,\ \tau))} \cdot \overline{c}_i(\xi',\ 1,\ \tau)]}_{rr_i(\overline{c}_i(\xi',\,1,\,\tau))} \cdot [\frac{1}{\overline{c}_i(\xi',\,1,\,\tau)} \frac{\partial\overline{c}_i(\xi',\,1,\,\tau)}{\partial\tau_d\left(\overline{\xi}\right)}]\right].$$

So

$$\boxed{\frac{\partial\overline{\pi}(\xi',\,\tau)}{\partial\tau_d\left(\overline{\xi}\right)} = \overline{\pi}(\xi',\tau) \cdot \left[rr_i(\overline{c}_i(\xi,\ 1,\ \tau)) \cdot g_{\overline{c}_i(\xi,\,1,\,\tau)} - rr_i(\overline{c}_i(\xi',\ 1,\ \tau)) \cdot g_{\overline{c}_i(\xi',\,1,\,\tau)}\right] \forall\overline{\xi} \in ET,} \tag{2}$$

where

$$g_{\overline{c}_i(\xi',\,1,\,\tau)} = \frac{1}{\overline{c}_i(\xi',\,1,\,\tau)} \cdot \frac{\partial \overline{c}_i(\xi',\,1,\,\tau)}{\partial \tau_d\left(\overline{\xi}\right)}\ \forall \xi' \in ET.\ \blacksquare$$

PROOF OF THEOREM 2.2.2: Note first that by assumption of the theorem

$$\frac{\partial \overline{c}_i(\xi',\,1,\,\tau)}{\partial \tau_d\left(\overline{\xi}\right)} = 0\ \forall \overline{\xi} \in ET \setminus ET\left(\xi'\right).$$

Therefore,

$$g_{\overline{c}_i(\xi',\,1,\,\tau)} = \frac{1}{\overline{c}_i(\xi',\,1,\,\tau)} \cdot \frac{\partial \overline{c}_i(\xi',\,1,\,\tau)}{\partial \tau_d\left(\overline{\xi}\right)} = 0\ \forall \overline{\xi} \in ET \setminus ET\left(\xi'\right).$$

We know that ξ is the initial node of the event tree ET. Thus, we van conclude by the previous Lemma 2.2.1. that

$$\boxed{\overline{\pi}(\xi',\,\tau) = b_i^{T(\xi')} \cdot \frac{u_i'(\overline{c}_i(\xi',\,1,\,\tau))}{u_i'(\overline{c}_i(\xi,\,1,\,\tau))} \cdot \Pr(\xi')\ \forall (\xi',\,i) \in ET^{+}(\xi) \times I} \tag{1}$$

and

$$\frac{\partial \overline{\pi}(\xi',\ \tau)}{\partial \tau_d\left(\xi\right)} = \overline{\pi}(\xi',\tau) \cdot rr_i(\overline{c}_i(\xi,\ 1,\ \tau)) \cdot g_{\overline{c}_i(\xi,\,1,\,\tau)}. \tag{2'}$$

Therefore,

$$\boxed{\overline{q}\left(\xi,\ \tau\right) = \sum_{\xi' \in ET^{+}(\xi)} b_i^{T(\xi')} \cdot \frac{u_i'(\overline{c}_i(\xi',\,1,\,\tau))}{u_i'(\overline{c}_i(\xi,\,1,\,\tau))} \cdot \Pr(\xi') \cdot \left(1 - \tau_d\left(\xi'\right)\right) \cdot d\left(\xi',\ \tau_d\right)\ \forall \xi \in ET.} \tag{3}$$

Also,

$$\begin{aligned} \frac{\partial \overline{q}(\xi,\,\tau)}{\partial \tau_d(\xi)} = & \underbrace{\sum_{\xi' \in ET^{+}(\xi)} \frac{\partial \overline{\pi}(\xi',\ \tau)}{\partial \tau_d\left(\xi\right)} \cdot \left(1 - \tau_d\left(\xi'\right)\right) \cdot d\left(\xi',\ \tau_d\right) +}_{Changes\ in\ the\ Stochastic\ Discount\ \overline{\pi}(\xi',\,\tau)\ Factor\ for\ \xi'} \\ & \underbrace{+ \sum_{\xi' \in ET^{+}(\xi)} \overline{\pi}(\xi',\ \tau) \cdot \left[\frac{\partial d\left(\xi',\ \tau_d\right)}{\partial \tau_d\left(\xi\right)} \cdot \left(1 - \tau_d\left(\xi'\right)\right) - d\left(\xi',\ \tau_d\right) \cdot \frac{\partial \tau_d\left(\xi'\right)}{\partial \tau_d\left(\xi\right)}\right]}_{Changes\ in\ After-tax\ Dividends\ (1-\tau_d(\xi')) \cdot d(\xi',\,\tau_d)} \forall \xi \in ET. \end{aligned}$$

Substituting (2') into the previous equation and taking into consideration that by the assumption of the Theorem

$$\frac{\partial \tau_d(\xi')}{\partial \tau_d(\xi)} = \frac{\partial d(\xi', \tau_d)}{\partial \tau_d(\xi)} = 0 \; \forall (\xi, \; \xi') \in ET \times [ET \setminus \{\xi\}] \, .$$

we get

$$\frac{\partial \overline{q}(\xi, \tau)}{\partial \tau_d(\xi)} = \sum_{\xi' \in ET^+(\xi)} rr_i(\overline{c}_i(\xi, 1, \tau)) \cdot g_{\overline{c}_i(\xi, 1, \tau)} \cdot \overline{\pi}(\xi', \tau) \cdot (1 - \tau_d(\xi')) \cdot d(\xi', \; \tau_d) =$$
$$= rr_i(\overline{c}_i(\xi, 1, \tau)) \cdot g_{\overline{c}_i(\xi, 1, \tau)} \cdot \sum_{\xi' \in ET^+(\xi)} \overline{\pi}(\xi', \tau) \cdot (1 - \tau_d(\xi')) \cdot d(\xi', \; \tau_d) =$$
$$= rr_i(\overline{c}_i(\xi, \; 1, \; \tau)) \cdot g_{\overline{c}_i(\xi, 1, \tau)} \cdot \overline{q}(\xi, \; \tau).$$

Therefore,

$$\boxed{\frac{\partial \overline{q}(\xi, \tau)}{\partial \tau_d(\xi)} = rr_i(\overline{c}_i(\xi, \; 1, \; \tau)) \cdot g_{\overline{c}_i(\xi, 1, \tau)} \cdot \overline{q}(\xi, \; \tau),} \tag{4}$$

where $rr_i(c) = -\left[\frac{u_i"(c) \cdot c}{u_i'(c)}\right]$ is the coefficient of relative risk aversion of an agent $i \in I$,

$$g_{\overline{c}_i(\xi, 1, \tau)} = \frac{1}{\overline{c}_i(\xi, 1, \tau)} \cdot \frac{\partial \overline{c}_i(\xi, 1, \tau)}{\partial \tau_d(\xi)}$$

and

$$\boxed{sign\left[\frac{\partial \overline{q}(\xi, \tau)}{\partial \tau_d(\xi)}\right] = sign\left[\frac{\partial \overline{c}_i(\xi, 1, \tau)}{\partial \tau_d(\xi)}\right] \; \forall (\xi, \; i) \in ET \times I.} \blacksquare$$

PROOF OF COROLLARY 2.2.3: Let the total supply of assets is given by $\overline{z} = \{\overline{z}(k)\}_{k \in K} \in \mathbb{R}_+^{|K|}$. Then,

$$\sum_{i \in I} \overline{c}_i(\xi, 1, \tau) =$$
$$\sum_{i \in I} (1 - \tau_{e_i}(\xi, \; 1)) \cdot e_i(\xi, 1, \tau_{e_i}) + \sum_{k \in K} (1 - \tau_d(\xi, \; k)) \cdot d(\xi, k, \tau_d) \cdot \overline{z}(k) \; \forall \xi \in ET.$$

Then, since all agents are identical we have

$$\overline{c}_i(\xi, 1, \tau) = (1 - \tau_e(\xi, \; 1)) \cdot e(\xi, 1, \tau_e) + \sum_{k \in K} \frac{\overline{z}(k)}{|I|} \cdot (1 - \tau_d(\xi, k)) \cdot d(\xi, k, \tau_d)$$

$\forall (\xi,\, i) \in ET \times I$.
All agents have zero initial endowments, i.e.,

$$e_i(\xi,\, 1,\, \tau_{e_i}) = 0 \;\forall\, (i,\ \xi) \in I \times ET.$$

Also, all assets' dividends are taxed identically, i.e.,

$$\tau_d(\xi,\, k) = \tau_d(\xi) \;\forall\, (\xi,\ k) \in ET \times K.$$

Thus, we have that

$$\overline{c}_i(\xi,\, 1,\, \tau) = (1 - \tau_d(\xi)) \cdot \sum_{k \in K} \tfrac{\overline{\overline{z}}(k)}{|I|} \cdot d(\xi,\, k,\, \tau_d) \;\forall (\xi,\, i) \in ET \times I.$$

Also, by the assumption of the Corollary

$$\tfrac{\partial d(\xi',\, \tau_d)}{\partial \tau_d(\xi)} = 0 \;\forall (\xi,\, \xi') \in ET \times ET.$$

Therefore,

$$\tfrac{\partial \overline{c}_i(\xi,\, 1,\, \tau)}{\partial \tau_d(\xi)} = -\sum_{k \in K} \tfrac{\overline{z}(k)}{|I|} \cdot d(\overline{\xi},\, k,\, \tau_d) \;\forall (\xi,\, i) \in ET \times I.$$

So we can conclude that

$$g_{\overline{c}_i(\xi,\, 1,\, \tau)} = \tfrac{1}{\overline{c}_i(\xi,\, 1,\, \tau)} \cdot \tfrac{\partial \overline{c}_i(\xi,\, 1,\, \tau)}{\partial \tau_d(\xi)} = \frac{-\sum_{k \in K} \tfrac{\overline{z}(k)}{|I|} \cdot d(\xi,\, k,\, \tau_d)}{(1-\tau_d(\xi)) \cdot \sum_{k \in K} \tfrac{\overline{z}(k)}{|I|} \cdot d(\xi,\, k,\, \tau_d)} = -\tfrac{1}{(1-\tau_d(\xi))} \;\forall \xi \in ET.$$

Therefore, by (4) we obtain

$$\tfrac{\partial \overline{q}(\xi,\, \tau)}{\partial \tau_d(\xi)} = -rr_i(\overline{c}_i\,(\xi,\ 1,\ \tau)) \cdot \tfrac{1}{(1-\tau_d(\xi))} \cdot \overline{q}\,(\xi,\ \tau) \;\forall (\xi,\, i) \in ET \times I.$$

Since all identical agents exhibit CRRA, i.e.,

$$rr_i(c) = -\left[\tfrac{u_i''(c) \cdot c}{u_i'(c)}\right] = a \;\forall i \in I,$$

we have that

$$\tfrac{\partial \overline{q}(\xi,\, \tau)}{\partial \tau_d(\xi)} = -a \cdot \tfrac{1}{(1-\tau_d(\xi))} \cdot \overline{q}\,(\xi,\ \tau) \;\forall \xi \in ET.$$

Hence,

$$E_{q(\xi,\,\tau),\,1-\tau_d(\xi)} = \frac{\frac{1}{\overline{q}(\xi,\,\tau)}\frac{\partial\overline{q}(\xi,\,\tau)}{\partial\tau_d(\xi)}}{\frac{1}{(1-\tau_d(\xi))}\frac{\partial(1-\tau_d(\xi))}{\partial\tau_d(\xi)}} = \frac{-a\cdot\frac{1}{(1-\tau_d(\xi))}}{-\frac{1}{(1-\tau_d(\xi))}} = a \;\forall\xi\in ET.$$

So

$$\boxed{E_{q(\xi,\,\tau),\,1-\tau_d(\xi)} = a\;\forall\xi\in ET.}$$

PROOF OF COROLLARY 2.2.4: We know that

$$R(\xi',\,\tau) = \frac{q(\xi',\,\tau)+d(\xi',\,\tau_d)}{q(\xi,\,\tau)},$$

where $\xi'\in\xi^+$. Differentiating $R(\xi',\,\tau)$ with respect to $\tau(\xi)$ we obtain

$$\frac{\partial R(\xi',\,\tau)}{\partial\tau_d(\xi)} = -\frac{q(\xi',\,\tau)+d(\xi',\,\tau_d)}{q^2(\xi,\,\tau)}\cdot\frac{\partial\overline{q}(\xi,\,\tau)}{\partial\tau_d(\xi)}.$$

We also know that

$$ATR(\xi',\,\tau) = \frac{q(\xi',\,\tau)+(1-\tau_d(\xi'))\cdot d(\xi',\,\tau_d)}{q(\xi,\,\tau)},$$

where $\xi'\in\xi^+$. Differentiating $ATR(\xi',\,\tau)$ with respect to $\tau(\xi)$ we obtain

$$\frac{\partial ATR(\xi',\,\tau)}{\partial\tau_d(\xi)} = -\frac{q(\xi',\,\tau)+(1-\tau_d(\xi'))\cdot d(\xi',\,\tau_d)}{q^2(\xi,\,\tau)}\cdot\frac{\partial\overline{q}(\xi,\,\tau)}{\partial\tau_d(\xi)}.$$

Therefore,

$$\boxed{sign\left[\frac{\partial R(\xi',\,\tau)}{\partial\tau_d(\xi)}\right] = sign\left[\frac{\partial ATR(\xi',\,\tau)}{\partial\tau_d(\xi)}\right] = -sign\left[\frac{\partial q(\xi,\,\tau)}{\partial\tau(\xi)}\right]\;\forall\,(\xi,\;\xi')\in ET\times\xi^+.}\;\blacksquare$$

PROOF OF THEOREM 2.2.5: Let ξ be the initial node of the event tree ET. Clearly,

$$\overline{\pi}(\xi',\,\tau) = b_i^{T(\xi')}\cdot\left(\frac{\overline{c}_i(\xi',\,1,\,\tau)}{\overline{c}_i(\xi,\,1,\,\tau)}\right)^{-a_i}\cdot\Pr(\xi')\;\forall(\xi',\,i)\in ET^+(\xi)\times I.$$

By the assumption of the Theorem

$$\frac{\overline{c}_i(\xi',\,1,\,\tau)}{\overline{c}_i(\xi,\,1,\,\tau)} = \frac{\sum\limits_{i\in I}\Big(1-\tau_{e_i}(\xi',\,1)\Big)\cdot e_i(\xi',\,1,\,\tau_{e_i})+\sum\limits_{k\in K}\overline{z}(k)\cdot(1-\tau_d(\xi',\,k))\cdot d(\xi',\,k,\,\tau_d)}{\sum\limits_{i\in I}\Big(1-\tau_{e_i}(\xi,\,1)\Big)\cdot e_i(\xi,\,1,\,\tau_{e_i})+\sum\limits_{k\in K}\overline{z}(k)\cdot(1-\tau_d(\xi,\,k))\cdot d(\xi,\,k,\,\tau_d)}$$

$\forall\,(\xi,\ \xi',\ i) \in ET \times ET \times I,$

where $\overline{z}(k)$ is the total number of outstanding shares of asset $k \in K$. Therefore,

$$\overline{\pi}(\xi',\,\tau) = b_i^{T(\xi')} \cdot \left(\frac{\sum\limits_{i\in I}\Big(1-\tau_{e_i}(\xi',\,1)\Big)\cdot e_i(\xi',\,1,\,\tau_{e_i}) + \sum\limits_{k\in K}\overline{z}(k)\cdot(1-\tau_d(\xi',\,k))\cdot d(\xi',\,k,\,\tau_d)}{\sum\limits_{i\in I}\Big(1-\tau_{e_i}(\xi,\,1)\Big)\cdot e_i(\xi,\,1,\,\tau_{e_i}) + \sum\limits_{k\in K}\overline{z}(k)\cdot(1-\tau_d(\xi,\,k))\cdot d(\xi,\,k,\,\tau_d)} \right)^{-a_i} \cdot \Pr(\xi')$$

$\forall(\xi',\,i) \in ET^+(\xi) \times I$ and and by equation (2)

$$\frac{\partial\overline{\pi}(\xi',\ \tau)}{\partial\tau_d\left(\overline{\xi}\right)} = \begin{cases} 0 & \forall\xi' \in ET^+(\xi) \setminus \{\overline{\xi}\} \\ -\overline{\pi}(\xi',\tau)\cdot a_i \cdot g_{\overline{c}_i(\xi',\,1,\,\tau)} & \text{if } \xi' = \overline{\xi} \end{cases} \qquad (2")$$

We also know that

$$\frac{\partial\overline{q}(\xi,\,\tau)}{\partial\tau_d\left(\overline{\xi}\right)} = \underbrace{\sum_{\xi'\in ET^+(\xi)} \frac{\partial\overline{\pi}(\xi',\ \tau)}{\partial\tau_d\left(\overline{\xi}\right)} \cdot \left(1-\tau_d\left(\xi'\right)\right)\cdot d\left(\xi',\ \tau_d\right)}_{Changes\ in\ the\ Stochastic\ Discount\ \overline{\pi}(\xi',\,\tau)\ Factor\ for\ \xi'} +$$

$$+\underbrace{\sum_{\xi'\in ET^+(\xi)} \overline{\pi}(\xi',\ \tau)\cdot\left[\frac{\partial d\left(\xi',\ \tau_d\right)}{\partial\tau_d\left(\overline{\xi}\right)}\cdot\left(1-\tau_d\left(\xi'\right)\right) - d\left(\xi',\ \tau_d\right)\cdot\frac{\partial\tau_d\left(\xi'\right)}{\partial\tau_d\left(\overline{\xi}\right)}\right]}_{Changes\ in\ After-tax\ Dividends\ (1-\tau_d(\xi'))\cdot d(\xi',\,\tau_d)} \ \forall\overline{\xi} \in ET^+(\xi).$$

Substituting (2") into the previous equation and taking into consideration that by the assumption of the Theorem

$$\frac{\partial\tau_d(\xi')}{\partial\tau_d(\xi)} = \frac{\partial d(\xi',\,\tau_d)}{\partial\tau_d(\xi)} = 0\ \forall\,(\xi,\ \xi') \in ET \times \left[ET \setminus \{\xi\}\right],$$

we get

$$\frac{\partial\overline{q}(\xi,\,\tau)}{\partial\tau_d\left(\overline{\xi}\right)} = -a_i\cdot\overline{\pi}(\overline{\xi},\,\tau)\cdot g_{\overline{c}_i(\overline{\xi},\,1,\,\tau)}\cdot\left(1-\tau_d\left(\overline{\xi}\right)\right)\cdot d\left(\overline{\xi},\ \tau_d\right) +$$
$$+\overline{\pi}(\overline{\xi},\,\tau)\cdot\left[\frac{\partial d\left(\overline{\xi},\,\tau_d\right)}{\partial\tau_d\left(\overline{\xi}\right)}\cdot\left(1-\tau_d\left(\overline{\xi}\right)\right) - d\left(\overline{\xi},\ \tau_d\right)\right]\ \forall\overline{\xi} \in ET^+(\xi).$$

Let

$$\frac{\partial\overline{q}(\xi,\,\tau)}{\partial\tau_d\left(\overline{\xi}\right)} > 0.$$

By assumption of the theorem we have

$$sign\left[\frac{\partial \overline{c}_i}{\partial \tau_d}\right] = -1 \ \forall i \in I.$$

Therefore,

$$a_i > \frac{d\left(\overline{\xi},\, \tau_d\right) - \frac{\partial d\left(\overline{\xi},\ \tau_d\right)}{\partial \tau_d\left(\overline{\xi}\right)} \cdot \left(1-\tau_d\left(\overline{\xi}\right)\right)}{-g_{\overline{c}_i(\overline{\xi},\, 1,\, \tau)} \cdot \left(1-\tau_d\left(\overline{\xi}\right)\right) \cdot d\left(\overline{\xi},\, \tau_d\right)}.$$

Thus,

$$\boxed{sign\left[\frac{\partial \overline{q}(\xi,\, \tau)}{\partial \tau_d\left(\overline{\xi}\right)}\right] = sign\left[a_i - B_i(\overline{\xi})\right],}$$

where

$$\boxed{B_i(\overline{\xi}) = \frac{d\left(\overline{\xi},\, \tau_d\right) - \frac{\partial d\left(\overline{\xi},\ \tau_d\right)}{\partial \tau_d\left(\overline{\xi}\right)} \cdot \left(1-\tau_d\left(\overline{\xi}\right)\right)}{-g_{\overline{c}_i(\overline{\xi},\, 1,\, \tau)} \cdot \left(1-\tau_d\left(\overline{\xi}\right)\right) \cdot d\left(\overline{\xi},\, \tau_d\right)} \ \forall \left(\overline{\xi},\ i\right) \in ET^{+}(\xi) \times I.} \blacksquare$$

PROOF OF THEOREM 2.2.6: Let the total supply of assets is given by $\overline{z} = \{\overline{z}(k)\}_{k \in K} \in \mathbb{R}_{+}^{|K|}$. Then,

$$\sum_{i \in I} \overline{c}_i(\xi,\, 1,\, \tau) =$$
$$\sum_{i \in I} (1 - \tau_{e_i}(\xi,\ 1)) \cdot e_i(\xi,\, 1,\, \tau_{e_i}) + \sum_{k \in K} (1 - \tau_d(\xi,\ k)) \cdot d(\xi,\, k,\, \tau_d) \cdot \overline{z}(k) \ \forall \xi \in ET.$$

Then, since all agents are identical we have

$$\overline{c}_i(\xi,\, 1,\, \tau) = (1 - \tau_e(\xi,\ 1)) \cdot e(\xi,\, 1,\, \tau_e) + \sum_{k \in K} \frac{\overline{z}(k)}{|I|} \cdot (1 - \tau_d(\xi,\, k)) \cdot d(\xi,\, k,\, \tau_d)$$

$\forall (\xi,\, i) \in ET \times I$.
Therefore,

$$\frac{\partial \overline{c}_i(\xi',\, 1,\, \tau)}{\partial \tau_d\left(\overline{\xi}\right)} = \begin{cases} 0 & \forall \xi' \in ET \setminus \left\{\overline{\xi}\right\} \\ \frac{\overline{z}}{|I|} \cdot \left[-d(\overline{\xi},\ \tau_d) + (1 - \tau_d(\overline{\xi})) \cdot \frac{\partial d\left(\overline{\xi},\, \tau_d\right)}{\partial \tau_d\left(\overline{\xi}\right)}\right] & \text{if } \xi' = \overline{\xi} \end{cases}.$$

So we can conclude that

$$g_{\overline{c}_i(\xi',\,1,\,\tau)} = \frac{1}{\overline{c}_i(\xi',\,1,\,\tau)} \cdot \frac{\partial \overline{c}_i(\xi',\,1,\,\tau)}{\partial \tau_d\left(\overline{\xi}\right)} = \begin{cases} 0 & \forall \xi' \in ET \setminus \left\{\overline{\xi}\right\} \\ \frac{\frac{\overline{z}}{|I|} \cdot \left[-d(\overline{\xi},\,\tau_d) + (1-\tau_d(\overline{\xi})) \cdot \frac{\partial d(\overline{\xi},\,\tau_d)}{\partial \tau_d(\overline{\xi})} \right]}{\overline{c}_i(\xi',\,1,\,\tau)} & \text{if } \xi' = \overline{\xi} \end{cases}.$$

and by equation (2)

$$\frac{\partial \overline{\pi}(\xi',\,\tau)}{\partial \tau_d\left(\overline{\xi}\right)} =$$

$$= \begin{cases} 0 & \forall \xi' \in ET \setminus \left\{\overline{\xi}\right\} \\ -\overline{\pi}(\xi',\,\tau) \cdot rr_i(\overline{c}_i(\xi',\,1,\,\tau)) \cdot g_{\overline{c}_i(\xi',\,1,\,\tau)} & \text{if } \xi' = \overline{\xi} \end{cases} \qquad (2\text{”’})$$

We also know that

$$\frac{\partial \overline{q}(\xi,\,\tau)}{\partial \tau_d\left(\overline{\xi}\right)} = \underbrace{\sum_{\xi' \in ET^+(\xi)} \frac{\partial \overline{\pi}(\xi',\,\tau)}{\partial \tau_d\left(\overline{\xi}\right)} \cdot \left(1 - \tau_d\left(\xi'\right)\right) \cdot d\left(\xi',\,\tau_d\right)}_{Changes\ in\ the\ Stochastic\ Discount\ \overline{\pi}(\xi',\,\tau)\ Factor\ for\ \xi'} +$$

$$+ \underbrace{\sum_{\xi' \in ET^+(\xi)} \overline{\pi}(\xi',\,\tau) \cdot \left[\frac{\partial d\left(\xi',\,\tau_d\right)}{\partial \tau_d\left(\overline{\xi}\right)} \cdot \left(1 - \tau_d\left(\xi'\right)\right) - d\left(\xi',\,\tau_d\right) \cdot \frac{\partial \tau_d\left(\xi'\right)}{\partial \tau_d\left(\overline{\xi}\right)} \right]}_{Changes\ in\ After-tax\ Dividends\ (1-\tau_d(\xi')) \cdot d(\xi',\,\tau_d)} \ \forall \overline{\xi} \in ET^+(\xi).$$

Substituting (2”’) into the previous equation and taking into consideration that by the assumption of the Theorem

$$\frac{\partial \tau_d(\xi')}{\partial \tau_d(\xi)} = \frac{\partial d(\xi',\,\tau_d)}{\partial \tau_d(\xi)} = 0 \ \forall \left(\xi,\ \xi'\right) \in ET \times \left[ET \setminus \{\xi\}\right],$$

we get

$$\frac{\partial \overline{q}(\xi,\,\tau)}{\partial \tau_d\left(\overline{\xi}\right)} = -rr(\overline{c}(\overline{\xi}, 1, \tau)) \cdot \overline{\pi}(\overline{\xi},\,\tau) \cdot g_{\overline{c}_i(\overline{\xi},\,1,\,\tau)} \cdot \left(1 - \tau_d\left(\overline{\xi}\right)\right) \cdot d\left(\overline{\xi},\ \tau_d\right) +$$
$$+ \overline{\pi}(\overline{\xi},\,\tau) \cdot \left[\frac{\partial d\left(\overline{\xi},\,\tau_d\right)}{\partial \tau_d\left(\overline{\xi}\right)} \cdot \left(1 - \tau_d\left(\overline{\xi}\right)\right) - d\left(\overline{\xi},\ \tau_d\right) \right] \forall \overline{\xi} \in ET^+(\xi).$$

Let

$$\frac{\partial \overline{q}(\xi,\,\tau)}{\partial \tau_d\left(\overline{\xi}\right)} > 0.$$

We know that

$$g_{\overline{c}_i(\overline{\xi},\, 1,\, \tau)} = \frac{1}{\overline{c}_i(\overline{\xi},\, 1,\, \tau)} \cdot \frac{\partial \overline{c}_i(\overline{\xi},\, 1,\, \tau)}{\partial \tau_d\left(\overline{\xi}\right)} < 0.$$

Therefore,

$$rr(\overline{c}(\overline{\xi},\, 1,\, \tau)) > \frac{d\left(\overline{\xi},\, \tau_d\right) - \frac{\partial d\left(\overline{\xi},\ \tau_d\right)}{\partial \tau_d\left(\overline{\xi}\right)} \cdot \left(1 - \tau_d\left(\overline{\xi}\right)\right)}{-g_{\overline{c}_i(\overline{\xi},\, 1,\, \tau)} \cdot \left(1 - \tau_d\left(\overline{\xi}\right)\right) \cdot d\left(\overline{\xi},\, \tau_d\right)}.$$

Thus,

$$\boxed{sign \left[\frac{\partial \overline{q}(\xi,\, \tau)}{\partial \tau_d\left(\overline{\xi}\right)}\right] = sign\left[rr(\overline{c}(\overline{\xi},\ 1,\ \tau)) - B(\overline{\xi})\right]\ \forall \overline{\xi} \in ET^{+}(\xi),}$$

where

$$\boxed{B(\overline{\xi}) = \frac{d\left(\overline{\xi},\, \tau_d\right) - \frac{\partial d\left(\overline{\xi},\ \tau_d\right)}{\partial \tau_d\left(\overline{\xi}\right)} \cdot \left(1 - \tau_d\left(\overline{\xi}\right)\right)}{-g_{\overline{c}_i(\overline{\xi},\, 1,\, \tau)} \cdot \left(1 - \tau_d\left(\overline{\xi}\right)\right) \cdot d\left(\overline{\xi},\, \tau_d\right)}\ \forall \overline{\xi} \in ET^{+}(\xi).} \blacksquare$$

PROOF OF COROLLARY 2.2.7: Let the total supply of assets is given by $\overline{z} = \{\overline{z}(k)\}_{k \in K} \in \mathbb{R}_{+}^{|K|}$. Then,

$$\sum_{i \in I} \overline{c}_i(\xi,\, 1,\, \tau) =$$
$$\sum_{i \in I} (1 - \tau_{e_i}(\xi,\ 1)) \cdot e_i(\xi,\, 1,\, \tau_{e_i}) + \sum_{k \in K} (1 - \tau_d(\xi,\ k)) \cdot d(\xi,\, k,\, \tau_d) \cdot \overline{z}(k)\ \forall \xi \in ET.$$

Then, since all agents are identical we have

$$\overline{c}_i(\xi,\, 1,\, \tau) = (1 - \tau_e(\xi,\ 1)) \cdot e(\xi,\, 1,\, \tau_e) + \sum_{k \in K} \frac{\overline{z}(k)}{|I|} \cdot (1 - \tau_d(\xi,\, k)) \cdot d(\xi,\, k,\, \tau_d)$$

$\forall (\xi,\, i) \in ET \times I$.
When all agents are the same and have zero initial endowments, it also means

$$e_i(\xi,\, 1,\, \tau_{e_i}) = 0\ \forall\, (i,\ \xi) \in I \times ET.$$

Also, all assets' dividends are taxed identically, i.e.,

$$\tau_d(\xi,\, k) = \tau_d(\xi)\ \forall\, (\xi,\ k) \in ET \times K.$$

Thus, we have that

$$\overline{c}_i(\xi,\,1,\,\tau)=\sum_{k\in K}\tfrac{\overline{z}(k)}{|I|}\cdot(1-\tau_d(\xi))\cdot d(\xi,\,k,\,\tau_d)\ \forall(\xi,\,i)\in ET\times I.$$

Also, by the assumption of the Corollary

$$\tfrac{\partial d}{\partial \tau_d}=0.$$

Therefore,

$$\tfrac{\partial \overline{c}_i(\overline{\xi},\,1,\,\tau)}{\partial \tau_d(\overline{\xi})}=-\sum_{k\in K}\tfrac{\overline{z}(k)}{|I|}\cdot d(\overline{\xi},\,k,\,\tau_d).$$

So we can conclude that

$$g_{\overline{c}_i(\overline{\xi},\,1,\,\tau)}=\tfrac{1}{\overline{c}_i(\overline{\xi},\,1,\,\tau)}\cdot\tfrac{\partial \overline{c}_i(\overline{\xi},\,1,\,\tau)}{\partial \tau_d\left(\overline{\xi}\right)}=\frac{-\sum\limits_{k\in K}\tfrac{\overline{z}(k)}{|I|}\cdot d(\overline{\xi},\,k,\,\tau_d)}{\sum\limits_{k\in K}\tfrac{\overline{z}(k)}{|I|}\cdot(1-\tau_d(\overline{\xi})\cdot d(\overline{\xi},\,k,\,\tau_d)}=-\tfrac{1}{(1-\tau_d(\overline{\xi})}.$$

Substituting $g_{\overline{c}_i(\overline{\xi},\,1,\,\tau)}$ into the expression for $B(\overline{\xi})$, we obtain

$$B(\overline{\xi})=\frac{d\left(\overline{\xi},\,\tau_d\right)}{\left[\tfrac{1}{(1-\tau_d(\overline{\xi})}\right]\cdot\left(1-\tau_d(\overline{\xi})\cdot d\left(\overline{\xi},\,\tau_d\right)\right.}=1.$$

Thus,

$$\boxed{sign\left[\tfrac{\partial \overline{q}(\xi,\,\tau)}{\partial \tau_d\left(\overline{\xi}\right)}\right]=sign\left[rr(\overline{c}(\overline{\xi},\ 1,\ \tau))-1\right]\ \forall\overline{\xi}\in ET^{+}(\xi).}\ \blacksquare$$

Proofs for Comparative Statics of FM Equilibria with Respect to the Endowment Tax τ_{e_i}

PROOF OF THEOREM 2.3.4: Let ξ be the initial node of the event tree ET. Clearly,

$$\overline{\pi}(\xi',\,\tau)=b_i^{T(\xi')}\cdot\left(\tfrac{\overline{c}_i(\xi',\,1,\,\tau)}{\overline{c}_i(\xi,\,1,\,\tau)}\right)^{-a_i}\cdot\Pr(\xi')\ \forall(\xi',\,i)\in ET^{+}(\xi)\times I.$$

By the assumption of the Theorem

$$\tfrac{\overline{c}_i(\xi',\,1,\,\tau)}{\overline{c}_i(\xi,\,1,\,\tau)}=\frac{\sum\limits_{i\in I}\left(1-\tau_{e_i}(\xi',\,1)\right)\cdot e_i(\xi',\,1,\,\tau_{e_i})+\sum\limits_{k\in K}\overline{z}(k)\cdot(1-\tau_d(\xi',\,k))\cdot d(\xi',\,k,\,\tau_d)}{\sum\limits_{i\in I}\left(1-\tau_{e_i}(\xi,\,1)\right)\cdot e_i(\xi,\,1,\,\tau_{e_i})+\sum\limits_{k\in K}\overline{z}(k)\cdot(1-\tau_d(\xi,\,k))\cdot d(\xi,\,k,\,\tau_d)}\ \forall\,(\xi,\ \xi')\in ET\times ET,$$

where $\overline{z}(k)$ is the total number of outstanding shares of asset $k \in K$. Therefore,

$$\overline{\pi}(\xi',\,\tau) = b_i^{T(\xi')} \cdot \left(\frac{\sum\limits_{i\in I}\left(1-\tau_{e_i}(\xi',\,1)\right)\cdot e_i(\xi',\,1,\,\tau_{e_i}) + \sum\limits_{k\in K}\overline{z}(k)\cdot(1-\tau_d(\xi',\,k))\cdot d(\xi',\,k,\,\tau_d)}{\sum\limits_{i\in I}\left(1-\tau_{e_i}(\xi,\,1)\right)\cdot e_i(\xi,\,1,\,\tau_{e_i}) + \sum\limits_{k\in K}\overline{z}(k)\cdot(1-\tau_d(\xi,\,k))\cdot d(\xi,\,k,\,\tau_d)} \right)^{-a_i} \cdot \Pr(\xi')$$

$\forall(\xi',\,i) \in ET^+(\xi) \times I$. Fix $\overline{\xi} \in ET^+(\xi)$. Then

$$\boxed{\frac{\partial\overline{\pi}(\xi',\,\tau)}{\partial\tau_{e_i}(\overline{\xi})} = \begin{cases} 0 & \forall\xi' \in ET^+(\xi) \setminus \left\{\overline{\xi}\right\} \\ -\overline{\pi}(\xi',\tau)\cdot a_i \cdot g_{\overline{c}_i(\xi',\,1,\,\tau)} & \text{if } \xi' = \overline{\xi} \end{cases}} \tag{5'}$$

We also know that

$$\frac{\partial\overline{q}(\xi,\,\tau)}{\partial\tau_{e_i}(\overline{\xi})} = \underbrace{\sum_{\xi'\in ET^+(\xi)} \frac{\partial\overline{\pi}(\xi',\,\tau)}{\partial\tau_{e_i}\left(\overline{\xi}\right)} \cdot \left(1-\tau_d\left(\xi'\right)\right) \cdot d\left(\xi',\,\tau_d\right) +}_{Changes\ in\ the\ Stochastic\ Discount\ \overline{\pi}(\xi',\,\tau)\ Factor\ for\ \xi'}$$

$$\underbrace{+\sum_{\xi'\in ET^+(\xi)} \overline{\pi}(\xi',\,\tau) \cdot \left[\frac{\partial d\left(\xi',\,\tau_d\right)}{\partial\tau_{e_i}\left(\overline{\xi}\right)} \cdot \left(1-\tau_d\left(\xi'\right)\right) - d\left(\xi',\,\tau_d\right) \cdot \frac{\partial\tau_d\left(\xi',\,\tau_d\right)}{\partial\tau_{e_i}\left(\overline{\xi}\right)} \right]}_{Changes\ in\ After-tax\ Dividends\ (1-\tau_d(\xi'))\cdot d(\xi',\,\tau_d)}$$

$\forall\overline{\xi} \in ET^+(\xi)$.

Substituting (5') into the previous equation and taking into consideration that by the assumption of the Theorem

$$\frac{\partial d}{\partial\tau_{e_i}} = \frac{\partial\tau_d}{\partial\tau_{e_i}} = 0,$$

we get

$$\boxed{\frac{\partial\overline{q}(\xi,\,\tau)}{\partial\tau_{e_i}(\overline{\xi})} = -a_i \cdot \overline{\pi}(\overline{\xi},\tau) \cdot g_{\overline{c}_i(\overline{\xi},\,1,\,\tau)} \cdot \left(1-\tau_d\left(\overline{\xi}\right)\right) \cdot d\left(\overline{\xi},\,\tau_d\right)}$$

and

$$\boxed{sign\left[\frac{\partial\overline{q}(\xi,\,\tau)}{\partial\tau_{e_i}(\overline{\xi})}\right] = -sign\left[\frac{\partial\overline{c}_i(\overline{\xi},\,1,\,\tau)}{\partial\tau_{e_i}(\overline{\xi})}\right] \quad \forall\overline{\xi} \in ET^+(\xi).\ \blacksquare}$$

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